WEIGHTED GRASSMANNIANS AND STABLE HYPERPLANE ARRANGEMENTS

VALERY ALEXEEV

ABSTRACT. We give a common generalization of (1) Hassett's weighted stable curves, and (2) Hacking-Keel-Tevelev's stable hyperplane arrangements.

CONTENTS

1. Introduction and main statements

The moduli space $\overline{\mathrm{M}}_{0,n}$ of stable n-pointed rational curves has many generalizations, beginning of course with $\overline{\mathrm{M}}_{g,n}$. For this paper, however, the following two generalizations will be important:

(1) Hassett's moduli $\overline{\mathrm{M}}_{0,\beta}$ of weighted stable n-pointed curves [Has03], and
(2) Hacking-Keel-Tevelev's moduli $\overline{\mathrm{M}}(r,n)$ of stable hyperplane arrangements [HKT06].

A *weight data*, or simply a *weight*, β is a collection of n of rational (or real) numbers $0 < b_i \leq 1$. We denote $\mathbb{1} = (1, \ldots, 1)$. A weighted stable curve of genus zero is a nodal curve $X = \cup \mathbb{P}^1$ whose dual graph is a tree, together with n points $B_1, \ldots, B_n$ satisfying two conditions:

(1) (on singularities) $B_i \neq$ the nodes, and whenever some points $\{B_i, \ i \in I\}$ coincide, one has $\sum_{i \in I} b_i \leq 1$.
(2) (numerical) $K_X + \sum b_i B_i$ is ample. In plain words, this means that for every irreducible component E of X, one has $|E \cap (X - E)| + \sum_{B_i \in E} b_i > 2$.

The space $\overline{\mathrm{M}}_{0,\beta}$ is the fine moduli space for flat families of such weighted curves; it is smooth and projective.

A *stable pair* $(X, B = \sum_{i=1}^{n} b_i B_i)$ is a natural higher-dimensional analogue of the above notion. It consists of a connected equidimensional projective variety X together with n Weil divisors B_i satisfying the following conditions (see [Ale06] for more details):

(1) (on singularities) X is reduced, and the pair (X, B) is slc (semi log canonical), and

(2) (numerical) $K_X + B$ is ample.

In [HKT06] the authors construct a projective scheme, which we will denote $\overline{M}(r, n)$, together with a flat family $f : (\mathcal{X}, \mathcal{B}_1, \ldots, \mathcal{B}_n) \to \overline{M}(r, n)$ such that every geometric fiber $(X, \sum B_i)$ is a stable pair in the above definition, with all coefficients $b_i = 1$. Over an open (but not dense in general) subset $M(r, n) \subset \overline{M}(r, n)$ this gives the universal family of n hyperplanes B_i on a projective space $X = \mathbb{P}^{r-1}$ such that B_i are in general position. The construction originates in [Kap93], see also [Laf03].

More generally, let β be a weight, and $B_1, \ldots, B_n$ be n hyperplanes in $\mathbb{P}^{r-1}$. Then the pair $(\mathbb{P}^{r-1}, \sum b_i B_i)$ is

(1) *lc (log canonical)* if for each intersection $\cap_{i \in I} B_i$ of codimension k, one has $\sum_{i \in I} b_i \le k$, and

(2) *klt (Kawamata log terminal)* if the inequalities are strict, in particular all $b_i < 1$.

(This is consistent with the standard definitions of the Minimal Model Program.)

The pair $(\mathbb{P}^{r-1}, \sum b_i B_i)$ is stable in the above definition iff it is lc (slc being an analog of lc for possibly nonnormal pairs) and $|\beta| = \sum_{i=1}^{n} b_i > r$. We call such pairs *weighted hyperplane arrangements*, or simply *lc hyperplane arrangements*. One easily constructs a fine moduli space $M_\beta(r, n)$ for them; it is smooth, of dimension $(r-1)(n-r-1)$, and usually not complete (but see Theorem 1.5 for the exceptions).

Throughout the paper, we work over an arbitrary commutative ring $\mathcal{A}$ with identity. The main results of this paper are the three theorems below and the detailed description of the weighted stable hyperplane arrangements given in Section 7.

Theorem 1.1 (Existence)**.** *For each r, n and a rational weight $\beta = (b_i)$ with $|\beta| = \sum b_i > r$, there exists a projective scheme $\overline{M}_\beta(r, n)$ together with a locally free (in particular, flat) family $f : (\mathcal{X}, \mathcal{B}_1, \ldots, \mathcal{B}_n) \to \overline{M}_\beta(r, n)$ such that:*

(1) *Every geometric fiber of f is an $(r-1)$-dimensional variety X together with n Weil divisors B_i such that the pair $(X, \sum b_i B_i)$ is stable.*

(2) *For distinct geometric points of $\overline{M}_\beta(r, n)$, the fibers are non-isomorphic.*

(3) *Over an open (but not dense in general) subset $M_\beta(r, n) \subset \overline{M}_\beta(r, n)$, f coincides with the universal family of weighted hyperplane arrangements.*

For every positive integer m such that all $mb_i \in \mathbb{N}$, the sheaf $\mathcal{O}_\mathcal{X}(m(K_\mathcal{X} + \sum b_i \mathcal{B}_i))$ is relatively ample and free over $\overline{M}_\beta$.

The fibers of f will be called *weighted stable hyperplane arrangements*, or simply *slc hyperplane arrangements*. As one has $M_{\mathbb{1}}(r, n) \subset M_\beta(r, n)$, in particular, each of $\overline{M}_\beta(r, n)$ provides a moduli compactification of the moduli space of generic hyperplane arrangements.

Definition 1.2. We define the *weight domain*

$$\mathcal{D}(r, n) = \{(b_i) \in \mathbb{Q}^n \mid 0 < b_i \le 1, \quad \sum b_i > r\}$$

and a subdivision of it into locally closed *chambers*, denoted $\mathrm{Ch}(\beta)$, by the hyper-planes $\sum_{i \in I} b_i = k$ for all $1 \le k \le r - 1$ and $I \subset \{1, \ldots, n\}$, and by the faces $b_i = 1$.

We introduce a partial order on the points of $\mathcal{D}(r, n)$: $\beta > \beta'$ if for all $1 \le i \le n$ one has $b_i \ge b_i'$, with at least one strict inequality.

We will frequently assume (r, n) fixed and drop it from the notation.

Example 1.3. $(r = 3,\ n = 5)$ Consider a 1-parameter family of 5 lines on $\mathbb{P}^2$ in general position such that in the limit B_1, B_2, B_5 meet at a point q_1, and B_3, B_4, B_5 meet at a point $q_2 \ne q_1$. This is not allowed by the lc singularity condition if $b_1 + b_2 + b_5 > 2$ or $b_3 + b_4 + b_5 > 2$. Since the spaces $\overline{\mathrm{M}}_\beta$ are proper, there is always a stable pair limit, but its shape depends on the weight:

(1) $\beta = (1, 1, 1, 1, 1 - \epsilon)$. The variety is $X_\beta = X^0 \cup X^1 \cup X^2$, where X^0 is the blowup of $\mathbb{P}^2$ at q_1 and q_2, and X^0 is glued along the exceptional $\mathbb{P}^1$'s to $X^1 = \mathbb{P}^2$, $X^2 = \mathbb{P}^2$. The divisor B_5 has three irreducible components, each of $B_1, \ldots, B_4$ has two. B_1, B_2 are contained in $X^0 \cup X^1$, and B_3, B_4 in $X^0 \cup X^2$. All five divisors are Cartier.

(2) $\beta' = \mathbb{1} = (1, \ldots, 1)$. The variety $X_{\beta'}$ is obtained from X_β by contracting the (-1)-curve $B_5 \cap X^0$. The image of X^0 is $X'^0 = \mathbb{P}^1 \times \mathbb{P}^1$, the divisors B_1, B_2 restricted to X'^0 are fibers of a ruling, and B_3, B_4 restricted to it are fibers of the second ruling. The divisor B_5 intersects X'^0 at one point, and so is not $\mathbb{Q}$-Cartier.

(3) $\beta'' = ((1 + \epsilon)/2, (1 + \epsilon)/2, 1, 1, 1 - \epsilon)$. The variety $X_{\beta''}$ is obtained from X_β by contracting X^1.

Note that $\beta' > \beta > \beta''$ and $\beta', \beta'' \in \overline{\mathrm{Ch}(\beta)}$, we have natural morphisms $X_{\beta'} \leftarrow X_\beta \to X_{\beta''}$, and the first of these morphisms is birational.

Theorem 1.4 (Reduction morphisms)**.**   (1) *(Same chamber) For β, β' lying in the same chamber, one has $\overline{\mathrm{M}}_\beta = \overline{\mathrm{M}}_{\beta'}$ and $(\mathcal{X}, \mathcal{B}_i)_\beta = (\mathcal{X}, \mathcal{B}_i)_{\beta'}$. In particular, the divisors $\sum (b_i - b_i) B_i$ are $\mathbb{Q}$-Cartier.*

(2) *For $\beta' \in \overline{\mathrm{Ch}(\beta)}$, there are natural reduction morphisms*

$$
\begin{array}{ccc}
(\mathcal{X}, \mathcal{B}_i)_\beta & \xrightarrow{\ \pi_{\beta,\beta'}\ } & (\mathcal{X}, \mathcal{B}_i)_{\beta'} \\
\downarrow & & \downarrow \\
\overline{\mathrm{M}}_\beta & \xrightarrow{\ \rho_{\beta,\beta'}\ } & \overline{\mathrm{M}}_{\beta'}
\end{array}
$$

One has

$$
\pi_{\beta,\beta'}^* \mathcal{O}_{\mathcal{X}_{\beta'}}(m(K_{\mathcal{X}_{\beta'}} + \sum b_i' \mathcal{B}_i)) = \mathcal{O}_{\mathcal{X}_\beta}(m(K_{\mathcal{X}_\beta} + \sum b_i' \mathcal{B}_i'))
$$

for any m such that all $m b_i' \in \mathbb{Z}$.

(3) *(Specializing up) For $\beta' \in \overline{\mathrm{Ch}(\beta)}$ with $\beta < \beta'$, $\rho_{\beta,\beta'}$ is an isomorphism, and on the fibers $\pi_{\beta,\beta'} : X \to X'$ is a birational contraction restricting to an isomorphism $X \setminus \cup B_i \xrightarrow{\sim} X' \setminus \cup B_i'$.*

(4) *For any $\beta > \beta'$, there is a natural reduction morphism $\rho_{\beta,\beta'} : \overline{\mathrm{M}}_\beta \to \overline{\mathrm{M}}_{\beta'}$. On the fibers, the rational map $\pi_{\beta,\beta'} : X \dashrightarrow X'$ is a sequence of log crepant contractions and log crepant birational extractions. Further, $X' = \mathrm{Proj} \oplus_{d \ge 0} H^0(X, \mathcal{O}(dm(K_X + \sum b_i' B_i)))$ is the log canonical model for the pair $(X, \sum b_i' B_i)$.*

Theorem 1.5 (Moduli for small weights). *Let $\alpha = (a_i)$ be a weight with $\sum a_i = r$ (lying on the boundary of $\mathcal{D}$) which belongs to the closure of a unique chamber $\mathrm{Ch}(\beta)$. Then*

$$\mathrm{M}_\beta = \overline{\mathrm{M}}_\beta = (\mathbb{P}^{r-1})^n //_\alpha PGL(r) = \mathrm{G}(r,n)//_\alpha \mathbb{G}_m^{n-1}$$

is the GIT quotient for the line bundle, resp. linearization corresponding to α.

For *any* boundary weight (i.e. with $|\alpha| = r$), we can formally define $\overline{\mathrm{M}}_\alpha$ to be the above GIT quotient. If α is not generic as in the theorem, then the points of $\overline{\mathrm{M}}_\alpha$ are lc hyperplane arrangements modulo a certain equivalence relation given by the $\mathbb{G}_m^n$-action.

Notations 1.6. We work over an arbitrary commutative base ring $\mathcal{A}$ with identity, without the Noetherian assumption, and indeed can work over any base scheme. A will denote an $\mathcal{A}$-algebra, and $k = \bar{k}$ an $\mathcal{A}$-algebra which is an algebraically closed field. The tilde will be used to denote affine schemes $\widetilde{X}$, cones $\widetilde{\Delta}$, etc., which are cones over the corresponding projective schemes X, polytopes Δ, etc.

It may help the reader to grasp some combinatorial aspects of this paper with the following general outline. The (unweighted) stable hyperplane arrangements are described by matroid tilings of the hypersimplex $\Delta(r,n)$. Their weighted counterparts are described by partial tilings of $\Delta(r,n)$ as viewed through a smaller "window" $\Delta_\beta(r,n)$; the window must be completely covered.

Another key idea is the GIT interpretation of the weight β explained in Section 6.

2. Matroid polytopes

We begin with some general definitions and then specialize them to the case of grassmannians.

Setup 2.1. We fix two lattices $\mathbb{Z}^N = \oplus \mathbb{Z}e_j$ and $\mathbb{Z}^n$, a homomorphism $\phi : \mathbb{Z}^N \to \mathbb{Z}^n$, and a homomorphism $\deg : \mathbb{Z}^n \to \mathbb{Z}$, such that $\deg \phi(e_j) = 1$ for all j. Associated to this data are affine $\mathbb{A} = \mathbb{A}^N$ and projective $\mathbb{P} = \mathbb{P}^{N-1}$ spaces over $\mathcal{A}$ and linear actions of split tori $\widetilde{T} = \mathbb{G}_m^n$ on $\mathbb{A}$ and of $T = \widetilde{T}/\operatorname{diag} \mathbb{G}_m$ on $\mathbb{P}$.

Let Δ be the lattice polytope that is the convex hull of $\phi(e_i)$, and $\widetilde{\Delta}$ be the corresponding cone in $\mathbb{R}^n$. We also fix a $\mathbb{Z}^n$-graded ideal $I[\widetilde{Z}] \subset \mathcal{A}[z_1, \ldots, z_N]$ such that the quotient is a locally free (i.e. projective) $\mathcal{A}$-module. Hence, $\widetilde{Z} \subset \mathbb{A}$ is a $\widetilde{T}$-invariant closed subscheme. Let $Z \subset \mathbb{P}$ be the corresponding T-invariant closed subscheme.

Definition 2.2. For a geometric point $p \in Z(k)$, the closure of the orbit $\overline{T.p}$ is a possibly nonnormal toric subvariety of $Z_k = Z \times_\mathcal{A} k$. It corresponds to a lattice polytope P which we will call the *Z-polytope* or the *moment polytope of p*. (Indeed, when $k = \mathbb{C}$, P is the moment polytope of $\overline{T.p}$, as defined in symplectic geometry.) A character $\chi \in \mathbb{Z}^n$ is in the cone $\widetilde{P}$ iff there exists a monomial $z^m = \prod_i^N z_i^{m_i}$ such that $\phi(m) = d\chi$ and $z^m(p) \neq 0$.

A *Z-tiling $\underline{P}$* is a face-fitting subdivision of Δ into Z-polytopes.

We fix several faces F_i, $i = 1, \ldots, n'$, of Δ. Each of them is defined by the inequality $l_i \leq 1$ for a unique $\mathbb{Z}$-primitive linear function $l_i(x_1, \ldots, x_n)$. In a completely parallel fashion with our grassmannian setup, an element $\beta = (b_i) \in \mathbb{Q}^{n'}$,

$b_i \leq 1$, is called a weight. For each weight we define a subpolytope

$$\Delta \supset \Delta_\beta = \{l_i \leq b_i\}$$

The *weight domain* $\mathcal{D} \subset \mathbb{R}^{n'}$ is the set of the weights for which Δ_β is nonempty and maximal-dimensional.

Definition 2.3. A *weighted Z-polytope* is a polytope of the form $P_\beta = P \cap \Delta_\beta$ for some Z-polytope P, called *the parent* of P_β, such that $\mathrm{Int}(P) \cap \Delta_\beta \neq \emptyset$.

A *weighted Z-tiling* $\underline{P}_\beta$ is a face-fitting tiling of $\Delta_\beta(r, n)$ by weighed Z-polytopes. The *partial* cover $\underline{P}$ of Δ by the parent polytopes is called *the parent cover* of $\underline{P}_\beta$.

Definition 2.4. The Z-chamber decomposition of $\mathcal{D}$ is defined as follows: β, β' lie in the same chamber if for every Z-polytope P, one has $P \cap \Delta_\beta \neq \emptyset \iff P \cap \Delta_{\beta'} \neq \emptyset$. Consequently, weighted Z-tilings of Δ_β and $\Delta_{\beta'}$ are in a bijection.

We now specialize these definitions to the case of the grassmannians. The polytope Δ in this case is called the *hypersimplex* and the Z-polytopes are called *matroid polytopes*. For the unweighted version, these notions were introduced in [GGMS87].

Let $\mathrm{G}(r, n)$ be the grassmannian of r-planes in a fixed affine space $\mathbb{A}^n$, together with its Plücker embedding into $\mathbb{P}(\wedge^r \mathbb{A}^n) = \mathbb{P}^{N-1}$, where $N = \binom{n}{r}$. Let $\widetilde{\mathrm{G}}(r, n) \subset \mathbb{A}^N$ be the affine cone. It is defined by the classical quadratic Plücker relations.

For $I = (i_1, \ldots, i_r)$, the Plücker coordinate p_I has character

$$\mathrm{wt}(p_I) = (1, \ldots, 1, 0 \ldots, 0)$$

with r ones in the places $i_1, \ldots, i_r$ and with $(n - r)$ zeros elsewhere.

Definition 2.5. The convex hull of these N points is called the *hypersimplex* $\Delta(r, n)$. Alternatively, it can be described as follows:

$$\Delta(r, n) = \left\{ (x_i) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \quad \sum_{i=1}^n x_i = r \right\}$$

It has $2n$ faces $F_i^+ = \{x_i = 1\}$ and $F_i^- = \{x_i = 0\}$, isomorphic to $\Delta(r-1, n-1)$ and $\Delta(r, n-1)$ respectively.

We fix the lattice $\Lambda \simeq \mathbb{Z}^n$ in $\mathbb{R}^n$ consisting of the vectors $(x_i) \in \mathbb{Z}^n$ such that $\sum x_i$ is divisible by r and a homomorphism $\deg : \Lambda \to \mathbb{Z}$ so that the characters of the Plücker coordinates p_I generate Λ and have degree 1.

Definition 2.6. A *matroid polytope* $P_V \subset \Delta(r, n)$ is the polytope corresponding to the toric variety $\overline{T.V}$ for some geometric point $[V \subset \mathbb{A}^n] \in \mathrm{G}(r, n)(k)$. (Theorem 2.8(1) implies that this projective toric variety and the corresponding affine variety are normal, unlike the case of general Z).

The equations of the coordinate hyperplanes restricted to V give n vectors $z_1, \ldots, z_n \in V^*$, what is called a *realizable matroid*. Then $\mathrm{wt}(p_{i_1, \ldots, i_r})$ is a vertex of P_V iff $z_{i_1}, \ldots, z_{i_r}$ form a basis of V^*. Alternatively, P_V can be described inside $\Delta(r, n)$ by the inequalities $\sum_{i \in I} x_i \leq \dim \mathrm{Span}(z_i, \ i \in I)$, for all $I \subset \{1, \ldots, n\}$.

One can also describe the matroid polytopes in terms of hyperplane arrangements. Let $\mathbb{P}V \simeq \mathbb{P}^{r-1}$ be the corresponding projective space and assume that it is not contained in any of the n coordinate hyperplanes H_i (i.e. all $z_i \neq 0$ on $\mathbb{P}V$); let $B_1, \ldots, B_n \subset \mathbb{P}V$ be $H_i \cap \mathbb{P}V$. Then $\mathrm{wt}(p_{i_1, \ldots, i_r})$ is a vertex of P_V iff $B_{i_1} \cap \ldots \cap B_{i_r}$ is empty. Alternatively, P_V can be described inside $\Delta(r, n)$ by the

inequalities $\sum_{i \in I} x_i \leq \mathrm{codim} \cap_{i \in I} B_i$ for all $I \subset \{1, \ldots, n\}$. Note that the matroid polytope in this case is not contained in any of the faces $F_i^- = \{x_i = 0\}$.

Definition 2.7. A *matroid tiling* $\underline{P}$ of $\Delta(r, n)$ is a face-fitting subdivision $\cup P(V_s)$ of $\Delta(r, n)$ into matroid polytopes.

Matroid polytopes form a very particular class of lattice polytopes, with many properties not shared by general lattice polytopes. Some of their properties can be summarized as follows:

Theorem 2.8.   (1) *Every matroid polytope is generating, i.e. its integral points generate the group of integral points of $\mathbb{R}P$. Moreover, the semigroup of integral points in $\widetilde{P}$ is generated by the vertices of P.*
   (2) *A matroid polytope of codimension c is in a canonical way the product of $c + 1$ maximal-dimensional matroid polytopes for smaller r, n. So, one has $r = r_0 + \ldots + r_c$ and $\{1, \ldots, n\} = I_0 \sqcup \cdots \sqcup I_c$, and $P = \prod P_j$, where $P_j \subset \Delta(r_j, |I_j|)$ is a maximal-dimensional matroid polytope.*

We now introduce the weighted versions of these notions.

Definition 2.9. Let $\beta = (b_1, \ldots, b_n)$ be a weight. A *weighted hypersimplex* is the polytope given by

$$\Delta_\beta(r, n) = \left\{ (x_i) \in \mathbb{R}^n \mid 0 \leq x_i \leq b_i, \quad \sum_{i=1}^n x_i = r \right\}.$$

Similarly, we also have definitions of a *weighted matroid polytope*, a *weighted matroid tiling* $\underline{P}_\beta$, and *the parent cover* of $\underline{P}_\beta$.

Question 2.10. Can every parent cover be extended to a *complete cover* of $\Delta(r, n)$? For $r = 2$ the answer is easily seen to be "yes". For $r \geq 3$ we expect the answer to be "no", following the general philosophy of "Mnev's universality theorem" (cf. [Laf03, Thm.I.14] which shows that matroid geometry can be arbitrarily complicated.

Theorem 2.11 (Chamber decomposition)**.** *The chamber decomposition of $\mathcal{D}(r, n)$ defined in 2.4 coincides with that of Definition 1.2.*

Proof. Starting with $\beta = \beta'$ and then varying the weight β, the matroid decompositions of Δ_β and $\Delta_{\beta'}$ may possibly change if for some matroid polytope $P \subset \Delta(r, n)$, a vertex of Δ_β would lie on a face of P that β' did not belong to.

Every vertex of Δ_β is of the following form: $x_i = 0$ or b_i, for all but possibly one i_0. Every face of a matroid polytope lies in the intersection of hyperplanes $\sum_{i \in I} x_i = k$ for some $1 \leq k \leq n - 1$ and $I \subset \{1, \ldots, n\}$. Possibly after replacing I by its complement, we can assume that $i_0 \notin I$. Then for some $J \subset I$ we get $\sum_{i \in J} b_i = k$. If β belongs to a face of P that β' did not belong to, then we get a new equation of this form. So β lies in a different, smaller chamber. $\square$

[GGMS87] gives three different interpretations of matroid polytopes. Here, we add another one.

Theorem 2.12. *The matroid polytope P_V is the set of points $(x_i) \in \mathbb{R}^n$ such that the pair $(\mathbb{P}V, \sum x_i B_i)$ is lc and $K_{\mathbb{P}V} + \sum x_i B_i = 0$; the interior $\mathrm{Int}\, P_V$ is the set of points such that $(\mathbb{P}V, \sum x_i B_i)$ is klt and $K_{\mathbb{P}V} + \sum x_i B_i = 0$.*

Proof. Indeed, the defining inequalities $\sum_{i\in I} x_i \leq \operatorname{codim} \cap_{i\in I} B_i$ of P_V also happen to be the conditions for the pair $(\mathbb{P}V, \sum x_i B_i)$ to be lc. Similarly with the strict inequalities and klt. $\qquad\square$

We don't even have to assume that $\mathbb{P}V \not\subset H_i = \{z_{i_0} = 0\}$ for this theorem: the pair $(\mathbb{P}V, \sum x_i B_i)$ can only be lc if $x_{i_0} = 0$, otherwise $\sum x_i B_i$ is not a divisor. And indeed if $\mathbb{P}V \subset H_i$ then $P_V \subset \{x_i = 0\}$, so the theorem still holds.

3. Moduli spaces for varieties with torus action

Let $Z \subset \mathbb{P}$ be a projective scheme locally free over $\mathcal{A}$ and invariant under the T-action, and $\widetilde{Z} \subset \mathbb{A}$ be its affine cone, with the $\widetilde{T}$-action, as in our general setup 2.1. Two moduli spaces of varieties with torus action will be relevant for this paper.

(1) The toric Hilbert scheme $\operatorname{Hilb}^{\widetilde{T}}(\widetilde{Z}, \widetilde{\Delta})$, constructed in [PS02, HS04].
(2) The moduli space $\mathrm{M}^T(Z, \Delta)$ of finite T-equivariant maps $Y \to Z$ of stable toric varieties Y over Z, constructed in [AB06], see also [AB04a, AB04b, Ale02]. This is the equivariant multiplicity-free version of the moduli space of branchvarieties [AK06].

Both of these moduli spaces are projective schemes. Both are available in much more general settings; we will only need the simplest versions.

For an $\mathcal{A}$-algebra A, $\operatorname{Hilb}^{\widetilde{T}}(\widetilde{Z}, \widetilde{\Delta})(A)$ is the set of closed $\widetilde{T}$-invariant subschemes $\widetilde{Y} \subset \widetilde{Z}_A = \widetilde{Z} \times_{\mathcal{A}} A$ which are *multiplicity-free:* for every $x \in \mathbb{Z}^n$ the graded piece $A[\widetilde{Y}]_x$ is a locally free rank-1 A-module if $x \in \widetilde{\Delta}$ and is 0 otherwise. A geometric fiber $\widetilde{Y}_k$ need not be reduced. $(\widetilde{Y}_k)_{\mathrm{red}}$ is a union of possibly non-normal toric varieties glued along torus orbits in a fairly complicated way.

In contrast, $\mathrm{M}^T(Z, \Delta)(A)$ is the set of locally free proper families Y over $\operatorname{Spec} A$ together with a finite T-equivariant morphism $f : Y \to Z_A$ such that every geometric fiber Y_k is a projective *stable toric variety*. In the ring $\oplus_{d\geq 0} H^0(Y, f^*\mathcal{O}(d))$, for each $x \in \mathbb{Z}^n$, the x-graded piece is a locally free rank-1 A-module if $x \in \widetilde{\Delta}$ and is 0 otherwise. A stable toric variety is a *reduced* variety glued from *normal* toric varieties along orbits in a very simple way, so that the result is seminormal. The price for such niceness is that $Y_k \to Z_k$ is a finite morphism rather than a closed embedding.

The projective stable toric variety Y_k comes with the polarization $L = f^*\mathcal{O}_Z(1)$. For each irreducible component of Y_k, this gives a lattice polytope P^s, and together they give a tiling $\Delta = \cup_{s\in \underline{P}} P^s$ describing the gluing in a rather precise way. If the cone semigroups $\widetilde{P}^s \cap \mathbb{Z}^n$ are generated in degree 1 then $Y_k \to Z_k$ is a closed embedding and gives a point of $\operatorname{Hilb}^{\widetilde{T}}(\widetilde{Z}, \widetilde{\Delta})$.

In general, $\mathrm{M}^T(Z, \Delta)$ is only a coarse moduli space since a finite map $Y_k \to Z_k$ may have automorphisms (deck transformations). However, it is a fine moduli space over an open subscheme where $Y_k \to Z_k$ is birational to its image (on every irreducible component).

Finally, we note how these moduli spaces change if we replace $\mathbb{P}$ by a d-tuple Veronese embedding, which means replacing the ring $\mathcal{A}[z_1, \ldots, z_N]$ by the subring generated by monomials of degrees divisible by d. The answer for $\operatorname{Hilb}^{\widetilde{T}}(\widetilde{Z})$ is very non-obvious, and sometimes they indeed change. The moduli space $\mathrm{M}^T(Z)$,

however, does not change. Indeed, the scheme Z does not change, and neither does the T-action. Therefore, $M^T(Z)$ can be defined as easily for a *rational polytope* Δ: it can always be rescaled to make it integral.

4. Review of the unweighted case

We now apply the general theory of the previous section to the grassmannians. Let $G = G(r, n)$ be the grassmannian with its Plücker embedding into $\mathbb{P}^N$, $N = \binom{n}{r}$, and $\widetilde{G}$ be its affine cone. Hence, $\mathcal{A}[\widetilde{G}]$ is generated by the N Plücker coordinates $p_{i_1, \ldots, i_r}$, modulo the usual quadratic relations. The corresponding polytope is precisely the hypersimplex $\Delta(r, n)$, and the polytopes P^s appearing in the constructions of the previous section are the matroid polytopes.

Definition 4.1 ([HKT06]). $\overline{M}(r, n) = \mathrm{Hilb}^{\widetilde{T}}(\widetilde{G}, \widetilde{\Delta})$.

A closed $\widetilde{T}$-invariant multiplicity-free subscheme $Y_k \subset G_k$ gives a matroid subdivision of $\Delta(r, n)$. 2.8(1) implies that Y_k is reduced and is a stable toric variety, so we are in the "nice case". Note that $\dim Y = n - 1 \neq r - 1$, so Y is not the required stable variety X. Instead, it should be thought of as the *log Albanese variety* $\log \mathrm{Alb}(X, B)$.

Definition 4.2 ([HKT06]). $X \subset Y$ is the intersection of Y with the subvariety in $G(r, n)$ defined by

$$G^e = \{V \subset \mathbb{A}^n \mid (1, 1, \ldots, 1) \in V\}.$$

G^e is a Schubert variety, isomorphic to $G(r - 1, n - 1)$.

One easily shows that $G^e \hookrightarrow G$ is a regular codimension $n - r$ embedding (not $\mathbb{G}_m^n$-equivariant), the zero set of a section of the tautological bundle Q on G. X does not contain any T-orbits. This implies that $X \subset Y$ is a regular codimension $n - r$ embedding as well, the zero section of the bundle $Q|_Y$ with $c_1(Q|_Y) = L$.

On the other hand, by [Ale02], the pair $(Y, \sum B_i^{\pm})$ has semi log canonical singularities and $K_Y + \sum B_i^{\pm} = 0$, where $B_i^{\pm}$ are the divisors corresponding the faces $F_i^{\pm}$ of $\Delta(r, n)$. In addition, $B_i^- \cap X = \emptyset$. Denoting $B_i^+|_X = B_i$ and combining this together gives the following:

Theorem 4.3 ([HKT06]).   (1) *There exists a smooth morphism $X \times T \to Y$ whose image is the open subset $Y \setminus \cup B_i^-$ swept by the T-orbits of $X \subset Y$; this is compatible with the divisors B_i and B_i^+.*
    (2) *Consequently, $(X, \sum B_i)$ and $(Y \setminus \cup B_i^-, \sum B_i^+ \setminus \cup B_i^-)$ are isomorphic locally in smooth topology; in particular, the pair $(X, \sum B_i)$ is slc.*
    (3) *$K_X + \sum B_i = (K_Y + \sum B_i^{\pm} + L)|_X = L|_X$; and so is ample.*
    (4) *The poset of the stratification of X defined by the irreducible components, their intersections, and the divisors B_i coincides with the poset of the stratification on $\Delta(r, n) \setminus \cup F_i^-$ defined by the tiling $\underline{P}$.*

5. Weighted grassmannians

Here we define certain projective schemes G_β and describe their basic properties. We begin with the elementary case which already contains the pertinent combinatorics of the general situation.

Let P' be a lattice polytope, and Y be the correponding projective toric scheme over $\mathcal{A}$ (a toric variety when working over a field k), together with an ample T-linearized ample invertible sheaf L'. Let m be a positive integer, $P = P'/m$ a rational polytope, and $L = L'/m \in \operatorname{Pic}(Y) \otimes \mathbb{Q}$ be the corresponding $\mathbb{Q}$-polarization.

Let us fix several faces F_i of this polytope. Each of them is given by a linear equation $x_i = b_i \in \mathbb{Q}$, where x_i is an integral primitive linear function, so that P lies in the half space $x_i \leq b_i$. Each of these faces corresponds to a divisor B_i on Y.

Now consider the polytope $P_{\beta'}$ obtained by replacing the inequalities $x_i \leq b_i$ with $x_i \leq b_i'$ for some $b_i' \in \mathbb{Q}$. Here, we denote $\beta = (b_i)$ and $\beta' = (b_i')$, so that $P = P_\beta$. Note that for some β' one may have $\dim P_{\beta'} < \dim P_\beta$.

One says that two polytopes are *normally equivalent* if their normal fans coincide, in other words, they define the same toric variety (with possibly different $\mathbb{Q}$-polarizations). The following elementary lemma is well-known, and we omit the proof.

Lemma 5.1.   (1) P_β and $P_{\beta'}$ are normally equivalent iff β' belongs to the interior $\operatorname{Ch}(\beta)$ of a certain rational polytope.
   (2) If $\beta' \in \overline{\operatorname{Ch}(\beta)}$ then there exists a natural morphism $\pi_{\beta,\beta'} : Y_\beta \to Y_{\beta'}$; it is birational if $\beta' > \beta$.
   (3) One has $\pi^*(L_{\beta'}) = L_\beta + \sum (b_i' - b_i) B_i$. Thus, any positive multiple of this $\mathbb{Q}$-line bundle that is integral, is semiample.

We note that this is precisely the kind of data that appears in Theorem 1.4. Now consider a projective scheme $Z \subset \mathbb{P}$ as in the general setup 2.1, $Z = \operatorname{Proj} \mathcal{A}[\widetilde{Z}]$.

Definition 5.2. For each weight β, let $\mathcal{A}[\widetilde{Z}_\beta] \subset \mathcal{A}[\widetilde{Z}]$ be the subalgebra generated by the monomials whose characters lie in $\widetilde{\Delta}_\beta$. We define $Z_\beta := \operatorname{Proj} \mathcal{A}[\widetilde{Z}_\beta]$.

Theorem 5.3.   (1) Every Z_β-polytope is the intersection of a Z-polytope with Δ_β.
   (2) There exists a rational map $Z \dashrightarrow Z_\beta$. It is regular on the open subset of points of Z whose moment polytopes intersect Δ_β.
   (3) Z and Z_β share an open subset Z_β^0 whose points are the points with moment polytopes intersecting $\operatorname{Int} \Delta_\beta$.
   (4) There exists a chamber decomposition of the weight domain into finitely many interiors of polytopes, with the following properties:
     (a) If β, β' belong to the same chamber, then $Z_\beta = Z_{\beta'}$.
     (b) If $\beta' \in \overline{\operatorname{Ch}(\beta)}$ then there exists a proper morphism $Z_\beta \to Z_{\beta'}$.
   (5) Further, assume that for any $\mathcal{A}$-field k, the corresponding variety Z_k is integral, normal, and its monomials span the whole $\widetilde{\Delta}$. Then $Z \dashrightarrow Z_\beta$ is a birational map, and in the previous statement one can take the Z-chamber decomposition defined in 2.4.

Proof. We reduce the proof to the elementary case 5.1 of toric schemes, as follows. The homomorphism $\phi : \mathbb{Z}^N \to \mathbb{Z}^n$ in the setup 2.1 gives a surjective map of polytopes $\sigma \to \Delta$, where σ is a simplex with N vertices. The preimage of Δ_β is a certain subpolytope $\sigma_\beta \subset \sigma$. Monomials of high enough degree d generate the subalgebra

$\mathcal{A}[\widetilde{Z}_\beta]^{(d)}$, and this gives the embedding of Z_β into the toric scheme corresponding to the polytope σ_β.

Now the properties (1–4) are elementary for the ambient toric schemes, and hence they also hold for the subschemes Z_β.

To prove (5), note that that (2) and (3) together imply that $Z \dashrightarrow Z_\beta$ is a birational map. Let β, β' belong to the same Z-chamber. Then on every geometric fiber we get a birational morphism $\varphi_k : (Z_\beta)_k \to (Z_{\beta'})_k$. Since the Z_β-polytopes and $Z_{\beta'}$-polytopes are the same, the T-orbits of $Z_\beta, Z_{\beta'}$ are in a dimension-preserving bijection, and so φ_k is finite. Since $(Z_{\beta'})_k$ is normal, φ_k is an isomorphism by the Main Zariski theorem. Since Z_β and $Z_{\beta'}$ are free over $\mathcal{A}$, and $\varphi : Z_\beta \to Z_{\beta'}$ is an isomorphism fiberwise, it is an isomorphism. $\square$

We now specialize to the case of grassmanians. Thus, for every weight $\beta \in \mathcal{D}(r, n)$ we get a projective scheme G_β, which we will call the *weighted grassmannian*, and the collection $\{G_\beta\}$ satisfies the conclusions of Theorem 5.3, where the chamber decomposition is the one defined in Definition 1.2.

6. **GIT theory of the universal family over the grassmanian**

A key role in our definition of weighted stable hyperplane arrangements will be played by the Geometric Invariant Theory of the universal family $U \to G(r, n)$. Let us first review the basics relevant to our case.

Let $Z \subset \mathbb{P}$ be as in the setup 2.1. Then we have an action of $T = \widetilde{T}/\operatorname{diag}(\mathbb{G}_m)$ on Z and an action of $\widetilde{T}$ on each $\mathcal{O}_Z(d)$, $d \in \mathbb{N}$. The character group of T is $\chi(T) = \{(x_i) \in \mathbb{Z}^n \mid \sum x_i = 0\}$.

A T-linearization of $\mathcal{O}_Z(d)$ is an extension of the T-action from Z to $\mathcal{O}_Z(d)$. It can be given by assigning to a monomial $z^m = z_1^{m_1} \dots z_n^{m_n}$ of degree $\sum m_i = d$ an element $\phi(m) \in \chi(T)$ so that $\phi(z^m) - \phi(z^{m'}) = m - m'$. This is equivalent to choosing an element $\beta \in \mathbb{Z}^n$ of degree d, so that $\phi(z^m) = m - \beta$. We can also take β to be of arbitrary positive degree, and subtract the unique element of degree d on the line $\mathbb{Q}\beta$. Then for every d' that is a multiple of d, the element of degree d' on this ray also describes the induced linearization of $\mathcal{O}_Z(d')$.

Given a T-linearized ample sheaf $L = \mathcal{O}_Z(d)$, one considers the ring of sections $R = R(Z, L) = \oplus_{d\geq0}\Gamma(Z, L^d)$. This ring was already graded by $\mathbb{Z}^n$ by the setup. The linearization provides a new grading by $\chi(T) = \mathbb{Z}^{n-1} \subset \chi(\widetilde{T}) = \mathbb{Z}^n$.

The GIT quotient $Z//_\beta T$ is defined to be $\operatorname{Proj} R_\beta$, where the latter denotes the elements of degree 0 in the $\chi(T)$-grading. In the original $\mathbb{Z}^n$-grading, this means that we consider the elements spanned by the monomials whose character in $\mathbb{Z}^n$ lies on the line $\mathbb{Q}\beta$. Note as well that replacing L by a positive power does not change $Z//_\beta T$. Hence, the input for this construction is a weight β up to a multiple, and an ample invertible sheaf L up to a multiple.

Applied to the grassmannian $G(r, n)$ and the Plücker line bundle $\mathcal{O}_G(1)$, this means that every weight $\beta \in \mathcal{D}(r, n)$ gives a linearization and a GIT quotient $G//_\beta T$. The quotients do not respect the chamber structure of $\mathcal{D}(r, n)$, however.

Our key observation now is that the chamber structure describes not the GIT quotients of G *but those of the universal family* U *over it.*

Let $U \subset \mathbb{P}^{n-1} \times G(r,n)$ be the universal family of linear spaces $\mathbb{P}V \subset \mathbb{P}^{n-1}$. Each of the n hyperplanes $H_i = \{z_i = 0\}$ in $\mathbb{P}^{n-1}$ defines a hyperplane $B_i \subset \mathbb{P}V$, unless $\mathbb{P}V \subset H_i$.

The natural ample invertible sheaves on U are $L_{a,b} = p_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(a) \otimes p_2^* \mathcal{O}_G(b)$ for $a, b \in \mathbb{N}$. The total ring of global sections of all of them is

$$\oplus_{a,b \geq 0} H^0(U, L_{a,b}) = \frac{\mathcal{A}[z_i, p_I]}{(\text{Plücker relations on } p_I, \ r_J)}$$

where for each $J = \{j_0, \ldots, j_r\}$, $r_J = \sum (-1)^k z_{j_k} p_{J \setminus j_k}$. The $\mathbb{Z}^n$-character of each z_i is e_i, the i-th coordinate vector in $\mathbb{Z}^n$, and for p_I it is $\sum_{i \in I} e_i$. Hence, the $\mathbb{Z}$-degrees of z_i and p_I are 1 and r respectively, differing slightly from 2.1.

Definition 6.1. We choose:
  (1) The ample $\mathbb{Q}$-line bundle $L_{a,b}$ with $(a,b) = (|\beta| - r, 1)$, or any actual ample invertible sheaf for a multiple (ma, mb) such that $m\beta$ is integral.
    Note that if $\mathbb{P}V \not\subset \cup H_i$ then $\mathcal{O}(K_{\mathbb{P}V} + \sum b_i B_i) = \mathcal{O}_{\mathbb{P}V}(|\beta| - r)$.
  (2) The T-linearization corresponding to β.

We denote the corresponding GIT quotient $U /\!/_\beta T$ by G_β^e.

Lemma 6.2. G_β^e is a closed subscheme of G_β.

Proof. Suppose that $m\beta$ is integral, and restrict to the subalgebra $\mathcal{A}[\widetilde{G}_\beta]^{(m)}$ whose homogeneous elements have degrees divisible by m. A monomial in p_I can be complemented to a monomial in p_I, z_i whose character is proportional to β exactly when its character, divided by the number of p_I's, lies in $\Delta(r,n)$ and in the cone $\beta - \mathbb{R}_{\geq 0}^n$. The intersection of these two sets is precisely Δ_β.

Hence, we have a surjective homomorphism $\mathcal{A}[\widetilde{G}_\beta]^{(m)} \to R_\beta^{(m)}$ sending a monomial $\prod p_I$ to its complement $z^s \prod p_I$. This gives the closed embedding. $\qquad \square$

Example 6.3. If $\beta = \mathbb{1}$ then every monomial in p_I can be complemented, and $G^e \subset G$ is the zero set of the equations $r_J(p_I, \mathbb{1})$. Thus, $G_{\mathbb{1}}^e$ is the same as G^e that appeared in Section 4.

GIT gives the description of $U /\!/_\beta T$ in terms of orbits for each geometric fiber. To recall, there are two open subsets in U:
  (1) The set U_β^{ss} of semistable points p for which there exists a section

$$s \in R_\beta = \oplus_{(a,b) \in \mathbb{Q}\beta} H^0(U, L_{a,b}) \text{ such that } s(p) \neq 0.$$

  (2) The set U_β^s of (properly) stable points whose orbit in U_β^{ss} is closed and the stabilizer is finite; in our case trivial. (This set was denoted by $U_{(0)}^s$ in [MFK94]. We use the currently prevalent notation.)

Then we have a surjective morphism $U_\beta^{ss} \to U /\!/_\beta T$, the action is free on U_β^s and U_β^s / T is a geometric quotient. Points of U_β^{ss} have the same image iff the closures of their orbits intersect. Among such orbits, there exists a unique closed one.

For the torus action, the Hilbert-Mumford's criterion for (semi)stability takes an especially simple form. The following criterion is well-known (e.g., cf. [BP90]):
  (1) $p \in U_\beta^{ss} \iff \beta$ belongs to the moment polytope of p.
  (2) $p \in U_\beta^s \iff \beta$ lies in the interior of the moment polytope of p and the latter is maximal-dimensional.

The moment polytope here lies in $\mathbb{R}^{n-1}$ which we shift so that it lies in the hyperplane $\sum x_i = |\beta|$ in $\mathbb{R}^n$. For our choice $L_{|\beta|-r,1}$ of an ample $\mathbb{Q}$-line bundle, the moment polytope of the point $[p \in \mathbb{P}V \subset \mathbb{P}^{n-1}] \in U$ is:

$$P_V + (|\beta| - r)\sigma_p, \quad \text{where}$$

(1) P_V, as before, is the matroid polytope of $[V \subset \mathbb{A}^n]$, and
(2) Denoting $I(p) = \{i \mid z_i(p) = 0\}$,

$$\sigma_p = \left\{ (x_i) \in \mathbb{R}^n \mid x_i \geq 0, \ \sum x_i = 1, \ \text{and } x_i = 0 \text{ for } i \in I(p) \right\}$$

Definition 6.4. Δ_β^p is the face (possibly empty) of Δ_β where $x_i = b_i$ for $i \in I(p)$.

Lemma 6.5.   (1) $p \in U_\beta^{\mathrm{ss}} \iff P_V \cap \Delta_\beta^p \neq \emptyset$.
   (2) $p \in U_\beta^{\mathrm{s}} \iff \operatorname{Int} P_V \cap \operatorname{Int} \Delta_\beta^p \neq \emptyset$ and $P_V + \Delta_\beta^p$ spans $\mathbb{R}^{n-1}$.

Proof. (1) $\beta \in P_V + (|\beta| - r)\sigma_p \iff P_V \cap \{\beta - (|\beta| - r)\sigma_p\} \neq \emptyset$. The intersection of the latter polytope with Δ is Δ_β^p.
   (2) The point is stable iff we can replace β with any nearby β'. This means that $\operatorname{Int} P_V \cap \operatorname{Int} \Delta_\beta^p \neq \emptyset$ and $P_V + \Delta_\beta^p$ spans $\mathbb{R}^{n-1}$. $\square$

Theorem 6.6.   (1) *If $P_V \cap \Delta_\beta = \emptyset$ or $V \subset \{z_i = 0\}$ for some i then no $p \in V$ is β-semistable.*
   (2) *Suppose $P_V \cap \Delta_\beta \neq \emptyset$. Then $p \in U_\beta^{\mathrm{ss}} \iff (\mathbb{P}V, \sum b_i B_i)$ is lc at p.*
   (3) *Suppose $P_V \cap \operatorname{Int} \Delta_\beta \neq \emptyset$. Then $p \in U_\beta^{\mathrm{s}} \iff (\mathbb{P}V, \sum b_i B_i)$ is klt at p.*

Proof. (1) If $P_V \cap \Delta_\beta = \emptyset$ then $p \notin U_\beta^{\mathrm{ss}}$ by the lemma. If $V \subset \{z_i = 0\}$ then $P_V \subset \{x_i = 0\}$, and $\Delta_\beta^p \subset \{x_i = b_i\}$, so they do not intersect.
   (2) Suppose $P_V \cap \Delta_\beta^p \neq \emptyset$. Take $\alpha = (a_i)$ in this intersection. By Theorem 2.12 the pair $(\mathbb{P}V, \sum a_i B_i)$ is lc. Since one has $\sum a_i B_i = \sum b_i B_i$ near p, the latter divisor is lc as well.
   Vice versa, assume that $(\mathbb{P}V, \sum b_i B_i)$ is lc at p. By assumption, there exists $\alpha \in P_V \cap \Delta_\beta$. If $\alpha \notin \Delta_\beta^p$ then we are going to construct another $\alpha' = (a_i') \in P_V \cap \Delta_\beta^p$.
   If P_V is maximal-dimensional and $\alpha \in \operatorname{Int} P_V$ then begin by increasing x_i for $i \in I(p)$ until we get to $x_i = b_i$ while decreasing x_i with $i \notin I(p)$ and keeping $x_i \geq 0$. This is possible to do since $\sum_{i \in I(p)} b_i \leq \operatorname{codim} \cap_{i \in I(p)} B_i \leq r - 1$. By doing this, we either achieve the required α' or get to a lower-dimensional matroid polytope $P_{V'}$. But by Theorem 2.8 $P_{V'}$ is the product of maximal-dimensional polytopes for lower (r_j, n_j). We finish by descending induction on r.
   (3) is proved by the same argument using the second part of Theorem 2.12. $\square$

Definition 6.7. Denote by $\mathbb{P}_\beta$ the projective toric scheme over $\mathcal{A}$ (a toric variety when working over k) corresponding to the polytope Δ_β.

In particular, $\mathbb{P}_{\mathbb{1}}$ is the toric variety corresponding to the hypersimplex $\Delta(r, n)$.

Theorem 6.8. *The morphism $U_\beta^{\mathrm{ss}} \to U /\!/_\beta T$ factors through $\mathbb{P}_\beta \times \mathrm{G}_\beta$.*

Proof. Consider $\mathbb{P}^{n-1} \times \mathrm{G}$ with the very ample sheaf $L_{a,b}$, $(a, b) \in \mathbb{Q}\beta$. The rational map $\mathbb{P}^{n-1} \times \mathrm{G} \dashrightarrow \mathbb{P}_\beta \times \mathrm{G}_\beta$ is given by the monomials $\prod p_I z^m$ whose character is proportional to β and, when normalized, the $\prod p_I$-part belongs to Δ_β and the z^m-part belongs to $\beta - \Delta_\beta$, which is just another copy of Δ_β, reflected.

This rational map is regular on the open subset where at least one of these monomials, considered as a section of L^d, is nonzero. But the ring generated by these monomials contains R_β, so this open subset contains U_β^{ss}. □

Recall from Section 5 that we denoted by G_β^0 an open subset of G_β and G that corresponds to $\mathrm{Int}\,\Delta_\beta$.

Definition 6.9. $U_\beta^0 \to \mathrm{G}_\beta^0$ will denote the pullback of $U^{\mathrm{ss}} \to \mathrm{G}_\beta$ under the open inclusion $\mathrm{G}_\beta^0 \to \mathrm{G}$.

Theorem 6.10.   (1) *The T-action on U_β^0 is free.*

(2) *The geometric quotient $\hat{\mathrm{G}}_\beta^e = U_\beta^0/T$ is projective and comes with a semi-ample invertible sheaf defining a proper birational morphism $\hat{\mathrm{G}}_\beta^e \to \mathrm{G}_\beta^e$, that is an isomorphism over $\mathrm{G}_\beta^e \cap \mathrm{G}_\beta^0$.*

Proof. (1) A point $V \in \mathrm{G}_\beta^0$ corresponds to a linear space $\mathbb{P}V \subset \mathbb{P}^{n-1}$ whose matroid polytope intersects $\mathrm{Int}\,\Delta_\beta$. Take $p \in V$ such that $p \in U_\beta^{\mathrm{ss}}$. Then by Theorem 6.6(1) the pair $(\mathbb{P}V, \sum b_i B_i)$ is lc at p. If we take a nearby $\beta' = (b_i')$ with $b_i' < b_i$ then $(\mathbb{P}V, \sum b_i B_i)$ will be klt. Then by 6.6(2) we have $p \in U_{\beta'}^{\mathrm{s}}$. Hence, $U_\beta^0 = U_{\beta'}^{\mathrm{s}}$, the action is free, and the quotient is projective.

By removing $(U_\beta^{\mathrm{ss}} \setminus U_\beta^0)$, we changed the equivalence relation on U_β^0: for some of the orbits in U_β^0 their closures in U_β^{ss} intersect, and so they map to the same point of $U/\!/_\beta T$. The criterion of Theorem 6.6 implies that every closed orbit in U_β^{ss} is contained in the closure of an orbit of U_β^0. Hence, $\hat{\mathrm{G}}_\beta^e \to \mathrm{G}_\beta^e$ is surjective. It is given by the pullback of an ample invertible sheaf $\mathcal{O}(m)$ on $\mathrm{Proj}\,R_\beta$. This morphism is an isomorphism on the open subset U_β^{s}/T which contains $\mathrm{G}_\beta^e \cap \mathrm{G}_\beta^0$. □

Remark 6.11. In the case of $\beta = \mathbb{1}$ our construction is different from that of [HKT06]. To explain it succinctly, [HKT06] proceeds "horizontally", while we proceed "vertically". The points in $U_{\mathbb{1}}^{\mathrm{ss}} \setminus U_{\mathbb{1}}^{\mathrm{s}}$ are: the points $p \in \cup B_i$ and the points $p \in V$ such that $P_V \subset \{x_i = 1\}$ for some i, satisfying the conditions of Theorem 6.6. So the action of T on $U_{\mathbb{1}}^{\mathrm{ss}}$ is not free. There are several ways to restrict it to a subset with a free action:

(1) "Horizontally", by removing the points $p \in \cup B_i$. The remaining set then is $U \cap (\mathbb{G}_m^{n-1} \times \mathrm{G})$, where $\mathbb{G}_m^{n-1} = \mathbb{P}^{n-1} \setminus \cup H_i$. This is the choice of [HKT06].
(2) "Vertically", by removing points with $P_V \subset \{x_i = 1\}$ – our choice.

7. Definitions of the moduli space and the family

Definition 7.1. (over $k = \bar{k}$) For each stable toric variety $Y \to \mathrm{G}_\beta$ over G_β, we define the corresponding *weighted stable hyperplane arrangement* as

$$X := (Y \times_{\mathrm{G}_\beta} U_\beta^{\mathrm{ss}})/\!/_\beta T.$$

We also define divisors $\bar{B}_i = (H_i \times \mathrm{G}_\beta) \cap U_\beta^{\mathrm{ss}}$ and then $B_i := (Y \times_{\mathrm{G}_\beta} \bar{B}_i)/\!/_\beta T$.

Theorem 7.2.   (1) *The T-action on the restriction to $Y \times_{\mathrm{G}_\beta^0} U_\beta^0$ is free. The geometric quotient $(\hat{X}, \hat{B}_i)$ by this free action is projective and comes with a semiample invertible sheaf defining a proper birational morphism $\hat{X} \to X$, an isomorphism on the complements of $\hat{B}_i$, B_i.*

(2) $X = Y \cap (preimage\ of\ G_\beta^e)$.

(3) X is reduced and B_i are reduced Weil divisors on X.

Proof. (1) follows immediately from Theorem 6.10 and (2) from Lemma 6.2 by functoriality of GIT quotients. (3) follows since GIT quotients of reduced schemes are reduced.   □

Example 7.3. Let $(\mathbb{P}V, \sum b_i B_i)$ be an lc hyperplane arrangement. Then $\Delta_\beta \subset P_V$. The weighted moment polytope of $(\mathbb{P}V, \sum b_i B_i)$ is therefore Δ_β itself. The normalization Y of the closure of the orbit $T.V$ is a toric variety and it comes with a finite morphism to G_β.

The pullback of G_β^0 under $Y \to G_\beta$ is simply the orbit $T.V$, isomorphic to T. Then $Y \times_{G_\beta} U_\beta^0 = \mathbb{P}V \times T$, and the quotient $\hat{X}$ is $\mathbb{P}V$ itself, together with the original divisors B_i. Since $\hat{X} \simeq \mathbb{P}^{r-1}$, the morphism $\hat{X} \to X$ must be an isomorphism. Hence, every lc hyperplane arrangement appears as a particular case of our construction.

Theorem 7.4.   (1) *Any weighted stable hyperplane arrangement $(X, \sum b_i B_i)$ is a stable pair, i.e. it has slc singularities and $K_X + \sum b_i B_i$ is an ample $\mathbb{Q}$-divisor.*

(2) *$\hat{X}$ is Gorenstein, and $\hat{B}_i$ are Cartier divisors.*

(3) *X is Cohen-Macaulay, and $X \setminus \cup B_i$ is Gorenstein.*

(4) *Assume $mb_i \in \mathbb{Z}$. Then $m(K_X + \sum b_i B_i)$ is the restriction under $Y \subset X$ of the ample invertible sheaf F_Y on Y corresponding to the polytope $m\Delta_\beta$.*

Proof. (1) By Theorems 2.12 and 6.6, $(U_\beta^0, \sum b_i \bar{B}_i^0)$ is a family of open lc subsets of hyperplane arrangements. Hence, U^0 is smooth, and there exists a finite sequence of blowups of $\mathbb{P}^{n-1}$ giving a simultaneous resolution of singularities of $(U_\beta^0, \sum b_i \widetilde{B}_i^0)$.

On the other hand, a stable toric variety Y together with its boundary is slc by [Ale02]; and the boundary is contained in $G_\beta \setminus G_\beta^0$ so can be omitted. This stable toric variety is Cohen-Macaulay, and its interior is Gorenstein by the Stanley-Reisner theory because the topological space Δ_β is a smooth manifold with boundary.

Therefore, the pullback $V := Y \times_{G_\beta} U_\beta^0$, together with the boundary, has slc singularities, is Gorenstein, and the divisors $\bar{B}_i$ are Cartier. Then the geometric quotient $\hat{X} = V/T$ by a free T-action is Gorenstein, and the divisors $\hat{B}_i$ are Cariter, giving (2).

Now let $m \in \mathbb{N}$ be such that $m\beta$ is integral, and let F_V, $F_{\hat{X}}$, F_X be the invertible sheaves on V, $\hat{X}$, X given by the GIT construction: F_V is the pullback of $L_{m\beta - mr, m}$, sections of F_V descend to sections of $F_{\hat{X}}$ and F_X, $F_{\hat{X}}$ is semiample and defines the contraction $\hat{X} \to X$, F_X is ample.

We observe that by construction one has $F_{\hat{X}} = \mathcal{O}_{\hat{X}}(m(K_{\hat{X}} + \sum b_i \hat{B}_i))$. This implies that $F_X = \mathcal{O}_X(m(K_X + \sum b_i B_i))$ and that $(X, \sum b_i B_i)$ is slc.

Since $X \setminus \cup B_i = \hat{X} \setminus \cup \hat{B}_i$, $X_i \setminus \cup B_i$ is Gorenstein. X is Cohen-Macaulay because it is the result of a log crepant contraction isomorphic outside of $\cup B_i$ and there exists a positive combination of B_i which is Cartier.

Let F_Y be the (integral) ample invertible sheaf corresponding to the polytope $m\Delta_\beta$. Then by the same argument as in Theorem 6.2 sections of F_Y restrict to sections of F_X. This gives (3).   □

Let $Y \to \mathrm{G}_\beta$ be a stable toric variety over G_β, and $\underline{P}_\beta = \{P_{V,\beta}\}$ be the corresponding cover of $\mathrm{Int}\,\Delta_\beta$. Each of these polytopes has a parent, so that $P_{V,\beta} = P_V \cap \Delta_\beta$.

We denote by $Y[P_{V,\beta}]$ the corresponding projective toric variety. We also denote by σ_n the simplex $\{(x_i) \in \mathbb{R}^n \mid x_i \geq 0,\ \sum x_i = r\}$. The corresponding to it toric variety is $\mathbb{P}^{n-1}$. If $P_{V,\beta}$ is maximal-dimensional then toric geometry gives a natural birational map $\psi[P_{V,\beta}] : \mathbb{P}^{n-1} \dashrightarrow Y[P_{V,\beta}]$, an isomorphism on the torus $\mathbb{G}_m^{n-1}$.

Now let $P_{V,\beta}$ be a weighted matroid polytope of codimension c, and let P_V be its parent, a matroid polytope. Recall from Theorem 2.8 that $P_V = \prod P_j$, the product of maximal-dimensional polytopes for a subdivision $\{1, \dots, n\} = \sqcup_{j=0}^c I_j$. Then we have a natural rational map $\mathbb{P}^{n-1} \dashrightarrow \prod \mathbb{P}^{|I_j|-1}$ which on an open subset is the quotient by a free action by $\mathbb{G}_m^c$.

In this case, we denote by $\psi[P_{V,\beta}] : \mathbb{P}^{n-1} \dashrightarrow Y[P_{V,\beta}]$ the latter rational map followed by the birational map $\prod \mathbb{P}^{|I_j|-1} \to Y[P_{V,\beta}]$ corresponding to polytopes $\prod \sigma_{|I_j|}$ and $P_{V,\beta}$.

Theorem 7.5. *The stratification of X into irreducible components and their intersections is in a dimension-preserving, with a shift by $n-r$, bijection with the tiling $\cup P_{V,\beta}$ of $\mathrm{Int}\,\Delta_\beta$. Moreover, the closure of the stratum corresponding to $P_{V,\beta}$ is the closure of the image of $\mathbb{P}V$ under $\psi[P_{V,\beta}] : \mathbb{P}^{n-1} \dashrightarrow Y[P_{V,\beta}]$. This rational map is regular on the open subset of $\mathbb{P}V$ where $(\mathbb{P}V, \sum b_i B_i)$ is lc. The image of this regular set gives a locally closed stratum in X corresponding to $P_{V,\beta}$.*

Proof. Each of the polytopes corresponds to an arrangement $\mathbb{P}V \subset \mathbb{P}^{n-1}$ so that the moment polytope P_V intersects $\mathrm{Int}\,\Delta_\beta$. Then we simply follow the T-orbit of $\mathbb{P}V$ in G_β^0, to the pullback in U_β^0, to the quotient in X, then back to the irreducible component of $Y[P_{V,\beta}]$ under the inclusion $X \subset Y$.

When $P_{V,\beta}$ is maximal-dimensional, the orbit $T.V$ in G_β^0 has trivial stabilizer. Hence, under the quotient by the free action by T, the open subset of lc points on $\mathbb{P}_V$ is preserved; then part is contracted by a birational morphism. If $\mathbb{P}_V$ has codimension c then the stabilizer of $T.V$ in G_β^0 is $\mathbb{G}_m^c$. Then the quotient factors through the quotient of an open subset of $\mathbb{P}^{n-1}$ by $\mathbb{G}_m^c$. $\qquad\square$

We are now ready to define the moduli space and the universal family of pairs over it.

Definition 7.6. $\overline{\mathrm{M}}_\beta(r, n) = \mathrm{M}^T(\mathrm{G}_\beta, \Delta_\beta)$.

Lemma 7.7. $\mathrm{M}^T(\mathrm{G}_\beta, \Delta_\beta)$ *is a fine moduli space.*

Proof. Indeed, every T-orbit in G_β^0 is also an orbit in G. If its stabilizer is finite then it is in fact trivial. Therefore, every irreducible component of a stable toric variety $Y \to \mathrm{G}_\beta$ maps to its image birationally, and the automorphism group of $Y \to \mathrm{G}_\beta$ is trivial. Then M^T is a fine moduli space. $\qquad\square$

Definition 7.8. The family of weighted stable hyperplanes arrangements $(\mathcal{X}, \mathcal{B}_i) \to \overline{\mathrm{M}}_\beta(r, n)$ is the GIT quotient of the pullback of $U_\beta^0 \to \mathrm{G}_\beta$ by the universal family of stable toric varieties $\mathcal{Y} \to \mathrm{M}^T(\mathrm{G}_\beta, \Delta_\beta)$.

Theorem 7.9. $(\mathcal{X}, \mathcal{B}_i) \to \overline{\mathrm{M}}_\beta(r, n)$ *is a locally free (in particular, flat) morphism.*

Proof. The families $U \to \mathrm{G}$ and $\mathcal{Y} \to \overline{\mathrm{M}}_\beta(r,n)$ are locally free, i.e. locally they are given by locally free modules. This implies that the pullback $\mathcal{Y} \times_{\overline{\mathrm{M}}} U^{\mathrm{ss}}_\beta$ is locally free. Algebraically, the GIT quotient is constructed by taking the degree-0 component in an algebra. Thus, this subalgebra is a direct summand, and a direct summand of a locally free module is locally free (by Kaplansky's theorem [Kap58], over any ring a module is locally free iff it is projective). $\qquad\square$

8. Completing the proofs of main theorems

Proof of Theorem 1.1 (Existence). The parts (1) and (3) were established in the previous section. The subset $\mathrm{M}_\beta(r,n) \subset \overline{\mathrm{M}}_\beta(r,n)$ is the open subset of $\mathrm{M}^T(\mathrm{G}_\beta, \Delta_\beta)$ where the stable toric varieties are irreducible, cf. Example 7.3. The sheaf $\mathcal{O}_\mathcal{X}(m(K_\mathcal{X} + \sum b_i B_i))$ is free over $\overline{\mathrm{M}}_\beta$ because by Theorem 7.4 it is the restriction of the invertible ample sheaf $F_\mathcal{Y}$ from the universal family of stable toric varieties that corresponds to the lattice polytope $m\Delta_\beta$, and $F_\mathcal{Y}$ is free: it's sections give the finite morphism to G_β.

The remaining part (2) is proved in the Reconstruction Theorem 8.1 below. $\qquad\square$

Proof of Theorem 1.4 (Reduction morphisms). (1) For β, β' in the same chamber, we have $\mathrm{G}_\beta = \mathrm{G}_{\beta'}$ by Theorem 5.3, applied to grassmannians. Also the conditions for GIT (semi)stability are the same. So the moduli and the families are the same.

(2) If $\beta' \in \overline{\mathrm{Ch}(\beta)}$, we have a reduction morphism $\mathrm{G}_\beta \to \mathrm{G}_{\beta'}$ again by Theorem 5.3. The third application of the same theorem gives the reduction morphism between the stable toric varieties $Y_\beta, Y_{\beta'}$ over $\mathrm{G}_\beta, \mathrm{G}_{\beta'}$. Finally, this gives in a canonical way the reduction morphisms between the pullbacks of the universal families and their GIT quotients.

Each $\pi_{\beta,\beta'}$ is log crepant. That is because the morphism on the ambient stable toric varieties is given by pullback of $L_\beta + \sum(b_i' - b_i)B_i$ (cf. 5.1, 5.3), and L_β on Y restricts to $K_X + \sum b_i B_i$ on X by Theorem 7.4.

(3) When specializing up, the morphism $\overline{\mathrm{M}}_\beta \to \overline{\mathrm{M}}_{\beta'}$ is an isomorphism. Indeed, a stable toric variety $Y \to \mathrm{G}_\beta$ is uniquely determined by its restriction Y^0 to G_β^0: Y is the partial normalization at the boundary of the closure of Y^0. But $\mathrm{G}_\beta^0 = \mathrm{G}_{\beta'}^0$ in this case.

Additionally, when specializing up, the morphism $X_\beta \to X_{\beta'}$ is simply our morphism $\hat{X} \to X$, so by Theorem 7.2 it is birational and an isomorphism outside $\hat{B}_i, B_i$.

(4) is an immediate consequence of the parts (1,2,3). $\qquad\square$

Note that if the source of the GIT quotient were fixed, with only the line bundle and the polarization changing, the statement would be an application of the well-known theory of variation of GIT quotients [BP90, Tha96, DH98].

Proof of Theorem 1.5 (Moduli for small weights). In this case $\mathrm{G}_\beta^0 = \mathrm{G}_\beta^{\mathrm{ss}} = \mathrm{G}_\beta^{\mathrm{s}}$, the T-action on it is free, and every stable toric variety over G_β is the closure of a unique T-orbit. Hence, $M^T(\mathrm{G}_\beta, \Delta_\beta) = \mathrm{G}_\beta^{\mathrm{s}}/T = \mathrm{G}/\!/_\beta T$.

The equivalence of the GIT quotients $(\mathbb{P}^{r-1})^n /\!/ \mathrm{PGL}(r)$ and $\mathrm{G}(r,n)/\!/T$ is well-known, see, e.g., [Kap93, 2.4.7]. $\qquad\square$

Intuitively, the contribution $p_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(|\beta| - r)$ to the polarization $L_{|\beta|-r,1}$ in this case approaches zero and only the quotient of G_β remains.

Theorem 8.1 (Reconstruction Theorem). *The stable toric variety $Y \to G_\beta$ can be uniquely reconstructed from $(X, \sum b_i B_i)$.*

Proof. Let $Y[P_{V,\beta}]$ be an irreducible component of a stable toric variety $Y \to G_\beta$, as in Theorem 7.5, and $X[P_{V,\beta}] \subset Y[P_{V,\beta}]$ be the corresponding irreducible component of X. We first show that $Y[P_{V,\beta}]$ can be reconstructed from $X[P_{V,\beta}]$ intrinsically.

Indeed, the boundary of $X[P_{V,\beta}]$ in X is labelled by the divisors B_i, some of them coinciding. Then the defining inequalities of $P_{V,\beta}$ can be read off this configuration: every missing intersection $\cap_{i \in I} B_i$ of codimension k gives the inequality $\sum_{i \in I} x_i \leq k$. This recovers the polytope $P_{V,\beta}$.

Then the embedding $X[P_{V,\beta}] \to Y[P_{V,\beta}]$ is recovered as follows. For every m such that $m\beta$ is integral, every integral point $(x_i) \in mP_{V,\beta}$ gives a section of the sheaf $\mathcal{O}_{X[P_{V,\beta}]}(m(K_X + \sum b_i B_i))$. Namely, it is a unique up to a constant section vanishing at B_i to the order x_i. The collection of these sections gives the embedding $X[P_{V,\beta}] \to Y[P_{V,\beta}]$ defined up to n choices of multiplicative constants, one for each B_i, i.e. up to the action of $\widetilde{T}$.

Finally, $\mathbb{P}V$ is recovered from the image of $X[P_{V,\beta}] \to Y[P_{V,\beta}]$ by applying the inverse of the rational map $\psi[P_{V,\beta}]$ of Theorem 7.5. Then the orbit $T.V$ in G_β^0 gives the morphism $Y[P_{V,\beta}] \to G_\beta$. The whole stable toric variety $Y \to G_\beta$ is recovered this way by looking at all maximal-dimensional polytopes $P_{V,\beta}$.   □

We note that for $\beta = \mathbb{1}$ this proof is very different from the one given in [HKT06], which does not extend to the weighted case.

9. **Some simple examples**

Example 9.1. $(r, n) = (2, 4)$, $\beta = \mathbb{1}$. Consider the subdivision of $\Delta(2, 4)$, the octahedron on the $\binom{4}{2}$ vertices ij, into two pyramids: P_1 missing the vertex 34, and P_2, missing the vertex 12.

P_1 corresponds to the configuration of 4 points in $\mathbb{P}^1$ for which the Plücker coordinate $p_{34} = 0$, i.e. $B_3 = B_4$. This polytope is given by the inequality $x_3 + x_4 \leq 1$, which is precisely the lc condition for this configuration. Similarly, for P_2 one has $B_1 = B_2$. On the intersection $P_1 \cap P_2$ one has $B_1 = B_2$, $B_3 = B_4$, and the defining inequalities become $x_1 + x_2 = 1$, $x_3 + x_4 = 1$, i.e. $P_1 \cap P_2 = \Delta(1, 2) \times \Delta(1, 2)$.

The irreducible component X_1 then is the closure of the image of $\mathbb{P}^1 \setminus B_3$, so isomorphic to $\mathbb{P}^1$; and similarly for X_2. The intersection $X_1 \cap X_2$ is the quotient $(\mathbb{P}^1 \setminus \{B_1, B_3\})/\mathbb{G}_m$, so a point. So X is a union of two $\mathbb{P}^1$'s intersecting at a point.

Example 9.2. $(r, n) = (2, 4)$, $\beta = (1/2, 1/2, 1, 1)$. Consider the trivial subdivision of Δ_β, with just the polytope itself. The points B_1 and B_2 may or may not coincide, depending on whether the parent polytope is the pyramid P_2 from the previous example, or $\Delta(2, 4)$, otherwise the points are pairwise distinct. $X = \mathbb{P}V = \mathbb{P}^1$.

Say, $B_1 = B_2$. Then $X \subset Y$ intersects the stratum corresponding to the edge $x_1 = x_2 = 1/2$ of Δ_β, at a point q. In this case, the T-translates of X do not sweep out an open subset of Y, and this is very different from the unweighted situation of Section 4. This may be the first indication that a GIT quotient must be involved.

If we consider the GIT quotient of the pullback family over the whole Y (not just $Y \cap G^0_\beta$ as in our construction), then on the boundary some fibers to the GIT quotient are modelled on the curve $\mathbb{A}^1 \cup_q \mathbb{A}^1$, which is a transversal slice of Y at the point q.

In all cases with $r = 2$ the considerations are quite similar, and produce a tree of $\mathbb{P}^1$'s.

Example 9.3. $(r, n) = (3, 5)$, this will correspond to Example 1.3. Begin with $\beta = \mathbb{1}$, and consider the subdivion of $\Delta(3, 5)$ into 3 polytopes: $P_0 = \{x_1 + x_2 + x_5 \leq 2, \ x_3 + x_4 + x_5 \leq 2\}$, $P_1 = \{x_1 + x_2 \leq 1\}$, and $P_2 = \{x_3 + x_4 \leq 1\}$.

Then P_0 corresponds to the configuration of 5 lines such that $B_1 \cap B_2 \cap B_5$ is a point, $B_3 \cap B_4 \cap B_5$ is a point, and otherwise generic. The matroid polytope P_V is obtained from $\Delta(3, 5)$ by cutting two corners, and the intersection $P_V \cap \{x_5 = 1\}$ has codimension 2, not 1 as might be expected: it is $\{x_5 = 1, x_1 + x_2 \leq 1, x_3 + x_4 \leq 1\}$, so the corresponding face gets contracted.

As in Theorem 7.5, the irreducible component X^0 is the image of $\mathbb{P}^2$ under $\mathbb{P}^4 \dashrightarrow Y[P_{V,\beta}]$. On $\mathbb{P}^2$ it blows up two points and contracts the strict preimage of B_5. The configuration $(\mathbb{P}V, \sum B_i)$ is lc outside of two points, so the divisor B_5 is present on $\hat{X}$; it is contracted by the log crepant morphism $\hat{X} \to X$.

For the weight $\beta = (1, 1, 1, 1, 1 - \epsilon)$, the face $P_V \cap \{x_5 = 1 - \epsilon\}$ has codimension 1, and the curve B_5 is not contracted.

Example 9.4. Consider the subdivision of Δ_β by a single hyperplane $x_1 + \cdots + x_{n_1} = r_1$, equivalently $x_{n_1+1} + \cdots + x_n = r_2$, with $r_1 + r_2 = r$, $n_1 + n_2 = n$. Then X is the union of $\mathrm{Bl}_{\mathbb{P}^{r_1-1}} \mathbb{P}^{r-1}$ and $\mathrm{Bl}_{\mathbb{P}^{r_2-1}} \mathbb{P}^{r-1}$ glued along $\mathbb{P}^{r_1-1} \times \mathbb{P}^{r_2-1}$.

Example 9.5. Let $(a_1, \ldots, a_{n-r+1}) \in \mathcal{D}(1, n-r+1)$ be a weight such that $\sum a_i > 1$ but $\sum_{i \in I} a_i \leq 1$ for any proper subset I. Let $\beta \in \mathcal{D}(r, n)$ be the weight consisting of α preceded by $(r - 1)$ 1's.

Then $\mathrm{M}_\beta = \overline{\mathrm{M}}_\beta = (\mathbb{P}^{n-r-1})^{r-1}$. For $r = 2$ this was established in [AG06, 4.5]. For the general case, we first observe that Theorem 1.5 applies in this case after replacing 1 with $1 - \epsilon$ for some $0 < \epsilon \ll 1$, and so $\overline{\mathrm{M}}_\beta$ a moduli space of lc hyperplane arrangements. The lc condition implies that the $(r - 1)$ hyperplanes with weight 1 must intersect normally. Restricting to an intersection to any $(r - 2)$ of these hyperplanes, a line, gives the $r = 2$ situation, for the weight $(1, a_i)$, and the moduli space for this is $\mathbb{P}^{n-r-1}$. Each of the hyperplanes with weight a_i is uniquely determined by the intersections with these $(r - 1)$ lines, and all of these configurations are lc. So $\mathrm{M}_\beta = (\mathbb{P}^{n-r-1})^{r-1}$.

Example 9.6. Let $\beta = (1, \ldots, 1, \epsilon, \ldots, \epsilon)$, $|\beta| = r + (n - r)\epsilon$. The case of $r = 2$ was introduced in [LM00], and $\overline{\mathrm{M}}_\beta(2, n)$ is the toric variety for the permutohedron, see also [AG06, 2.11(4)].

For any r, the closure of M_β in $\overline{\mathrm{M}}_\beta$ is the toric variety for the fiber polytope $\Sigma(\sigma_r^{\oplus(n-r)} \to (n - r)\sigma_r)$, where σ_r is the simplex with r vertices and side 1, and $(n - r)\sigma_r$ is σ_r dilated by $(n - r)$.

This moduli space also has an interpretation as the moduli space of stable toric pairs $(X, D_1, \ldots, D_{n-r})$, as in [Ale02] but with $(n - r)$ divisors instead of one. Explaining this in detail would take quite some space, and is better done elsewhere.

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