A NOTE ON TERMINATION OF FLIPS

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The purpose of this short note is to discuss the following

Key Lemma 0.1. *Let*

$$(Y^0, B^0) \dashrightarrow (Y^1, B^1) \dashrightarrow (Y^2, B^2) \dashrightarrow \dots$$

be a sequence of birational transformations such that

(1) *Each* $(Y^n, B^n = \sum_{i=1}^N b_i^n B_i^n)$ *is a* $\mathbb{Q}$*-factorial terminal pair with* $0 < b_i^n < 1$.
(2) *Each*

$$Y^n \xrightarrow{\psi_{n+1}^-} W^{n+1} \xleftarrow{\psi_{n+1}^+} Y^{n+1}$$

is a flip with respect to the divisor B^n.
(3) B_i^n *is a birational transform of* B_i^0;
(4) $B^{n+1} \le B^n$, *i.e.* $b_i^{n+1} \le b_i^n$ *for all* n *and* i.

Fix a real number $s \in [0, 1)$. *Then a valuation* v *with the center* $C(v, Y^{n+1})$ *in the flipped locus and discrepancy*

$$a(v; Y^{n+1}, B^n) = 1 - s \quad \text{or} \quad a(v; Y^{n+1}, B^{n+1}) = 1 - s$$

cannot appear infinitely many times.

As explained in [Kaw03], termination of 4-dimensional klt flips follows from this Lemma by using backtracking and crepant descent, see Section 2 below. We were unable to prove Key Lemma in full generality. Here, we establish it if one of the following addional conditions holds:

(1) either $\inf b_i^n > 0$,
(2) or $s \ne 0$.

In particular, this provides a new proof for termination of terminal (and canonical) 4-dimensional pairs with constant B^n, which is the main result of [Fuj04]; it also gives an effective bound on the number of $(1,2)$- and $(2,2)$-flips.

We use the customary notations. In particular, for a discrete rank 1 valuation v of the function field of Y, $a(v; Y, B)$ or simply a_v denotes the discrepancy (not the log discrepancy!) of v with respect to the $\mathbb{Q}$-Cartier divisor $K_Y + B$. The center of a valuation v on a variety W will be denoted by $C(v, W)$. For a flip

$$Y^- \xrightarrow{\psi^-} W \xleftarrow{\psi^+} Y^+$$

the set $\operatorname{Exc}\psi^-$ is called the flipping locus and $\operatorname{Exc}\psi^+$ the flipped locus.

For a complex projective variety D, we denote by $\rho(D)$ the dimension of subspace of $H_{2(\dim D-1)}(D, \mathbb{R})$ generated by algebraic cycles. $\nu : \widetilde{D} \to D$ will denote the normalization of of D.

Whenever two varieties are isomorphic in codimension 1, we may denote birational transforms of divisors by the same letters.

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1. Creative counting

Our *difficulty* will be a real number $d_{s,f}(W;Y,B)$ which will depend on

(1) a real number $s \in [0,1)$,
(2) a normal variety W on which the centers of valuations will be considered,
(3) a terminal pair $(Y, B=\sum b_i B_i)$ such that $0<b_i<1$, and such that Y and W are isomorphic in codimension 1. Discrepancies will be computed with respect to the divisor K_Y+B,
(4) a decreasing function $f:(0,+\infty)\to\mathbb{R}_{\ge 0}$ satisfying two conditions:
   (a) $f(x)>0$ for $x<1-s$ and $f(x)=0$ for $x\ge 1$.
   (b) Whenever $0<\sum m_i b_i<1$ for some $m_i\in\mathbb{Z}_{\ge 0}$, one has
$$f(1-\sum m_i b_i) - \sum m_i f(1-b_i) \ge 0.$$

For example, one can take $f(x)=1-x$ for $x<1-s$ and $f(x)=0$ for $x\ge 1-s$. We will define another function $g:(0,1)\to\mathbb{R}_{\ge 0}$ by the formula
$$g(b)=\sum_{k=1}^{\infty} f\big(k(1-b)\big)$$
It easily follows that there are only finitely many nonzero terms in this sum and that g is an increasing function. The meaning of these conditions will be clear from the proof of the next two lemmas.

Let $\nu : \coprod \widetilde{B}_i \to \cup B_i$ be the normalization of the divisor $\operatorname{Supp} B$. Then the preimage $\widetilde{C}=\nu^{-1}(C)$ splits into the union of irreducible components with $\widetilde{C}_{i,\alpha}\subset \widetilde{B}_i$.

Definition 1.1.
$$d_{s,f}(W;Y,B)= \sum g(b_i)\rho(\widetilde{B}_i) + \sum_{v;\ \operatorname{codim} C(v,W)>2} f(a_v) + \sum_{\text{irr.} C\subset W; \operatorname{codim} C=2} \left[\sum_{v;\ C(v,W)=C} f(a_v) - \sum_{\widetilde{C}_{i,\alpha}} g(b_i)\right]$$

We set $d_{s,f}(Y,B) := d_{s,f}(Y;Y,B)$.

Remark 1.2. For $f(x)=\max(1-x,0)$ this definition is a version of Shokurov's "stringy" invariant ρ^2 in [Sho03] except that we take into account higher "echos" (which is necessary).

Lemma 1.3. *$d_{s,f}(W;Y,B)$ is well defined, i.e. only finitely many summands are nonzero.*

Proof. Let C be a codimension-2 subvariety of Y whose generic point lies in $B\setminus \operatorname{Sing} B$, namely on a component B_i. The only possible valuations with $C(v,Y)=C$ and $a(v;Y,B)<1$ are obtained as follows. The divisor E_1 corresponds to the blowup of C (minus the singularities), E_2 to the blowup the intersection of E_1 with the strict preimage of B_i, E_3 to the blowup the intersection of E_2 with the strict preimage of B_i etc.

We will call E_k the *k-th echo* of B_i along C. The discrepancy of the corresponding valuation is $a_v = k(1-b_i)$. There are infinitely many codimension-2 subvarieties C of $B \setminus \operatorname{Sing} B$ but according to our definition of $g(b)$ the corresponding summand in $d_{s,f}$ is zero.

It is well known that there are only finitely many non-echo valuations of a terminal pair (Y, B) with $a_v < 1$. This proves that $d_{s,f}(Y; Y, B)$ is well-defined. Finally, W and Y differ in finitely many codimension-2 subvarieties, so $d_{s,f}(W; Y, B)$ is also well-defined. ☐

Lemma 1.4. $d_{s,f}(Y, B) \geq 0$.

Proof. Clearly, $\sum g(b_i)\rho(\widetilde{B}_i) \geq 0$, so we only need to prove that the contributions from codimension-2 subvarieties are nonnegative. This is a computation that can be done assuming that B_i's are curves on a nonsingular surface Y, intersecting at a point C. Moreover, we can and will treat analytic branches of B_i's as different divisors.

Let b_1 be a maximal coeffient among b_i. Note that $b_i < 1/2$ for $i > 1$ because the pair (Y, B) is terminal. Therefore, for $i > 1$ one has $g(b_i) = f(1 - b_i)$. The discrepancy of the exceptional divisor E_1 of the first blowup is $1 - \sum m_i b_i$. Inductively, the discrepancy of the exceptional divisor E_k obtained by blowing up the intersection of E_{k-1} and the strict transform of B_1 is $\leq 1 - \sum m_i b_i + (k-1)(1-b_1)$. Therefore:

$$\sum_{v;\ C(v,W)=C} f(a_v) - \sum_{\widetilde{C}_{i,\alpha}} g(b_i) \geq$$

$$\begin{aligned} &\geq f(1 - \sum m_i b_i) + \sum_{k=1}^{\infty} f(1 - kb_1 - \sum m_i b_i) - \sum m_i g(b_i) = \\ &= f(1 - \sum m_i b_i) - \sum m_i f(1 - b_i) + \\ &+ \sum_{k=2}^{\infty} f\Big(1 - \sum m_i b_i + (k-1)(1-b_1)\Big) - f\big(k(1-b_1)\big) \end{aligned}$$

The first half of the last expression is nonnegative by our condition (4b), and the second half is nonnegative because f is decreasing. ☐

Lemma 1.5. *Suppose $\psi : Y \to W$ is a birational morphism which is an isomorphism in codimension 1. Then*

$$d_{s,f}(Y; Y, B) = d_{s,f}(W; Y, B)$$

Proof. The expression $\sum g(b_i)\rho(\widetilde{B}_i)$ drops by $\sum_{\widetilde{C}_{i,\alpha}} g(b_i)$, where the sum goes over codimension-2 subvarieties of Y such that $\operatorname{codim} \psi(C) > 2$. This is true because on a normal variety exceptional divisors are linearly independent in homology. The last part of $d_{s,f}$ increases by the same amount. So they cancel out. ☐

For the next statement, we use the setup of Key Lemma.

Lemma 1.6. (1) $d_{s,f}(Y^n, B^n) \geq d_{s,f}(Y^{n+1}, B^n)$ *and the strict inequality holds if there exists a valuation with the center $C(v, Y^{n+1})$ in the flipped locus and $a(v; Y^{n+1}, B^n) = 1 - s$.*

(2) *There exists n_0 such that for $n \geq n_0$, $d_{s,f}(Y^{n+1}, B^n) \geq d_{s,f}(Y^{n+1}, B^{n+1})$ and the strict inequality holds if there exists a valuation with the center $C(v, Y^{n+1})$ in the flipped locus and $a(v; Y^{n+1}, B^{n+1}) = 1 - s$.*

(3) *If there exists $\epsilon > 0$ such that $f(x) \geq \epsilon$ for all $x < 1 - s$ and $f(a) = 0$ for $x \geq 1 - s$ then Key Lemma follows.*

Proof. (1) By the last lemma, we have

$$\begin{aligned} d_{s,f}(Y^n, B^n) &= d_{s,f}(W^{n+1}; Y^n, B^n) \\ d_{s,f}(Y^{n+1}, B^n) &= d_{s,f}(W^{n+1}; Y^{n+1}, B^n) \end{aligned}$$

On the other hand, one has

$$d_{s,f}(W^{n+1}; Y^n, B^n) \geq d_{s,f}(W^{n+1}; Y^{n+1}, B^n)$$

because $a(v; Y^n, B^n) \leq a(v; Y^{n+1}, B^n)$ by the basic property of flips. If there is a valuation with $a = 1 - s$ then one of the terms $f(a_v)$ in $d_{s,f}$ dissapears and the inequality becomes strict.

(2) follows similarly because $a(v; Y^{n+1}, B^{n+1}) \leq a(v; Y^{n+1}, B^n)$. The condition $n \geq n_0$ is needed to make sure that the same number of echos is counted in the definition of $g(b)$, i.e. that

$$k(1 - b_i^n) < 1 - s \quad \Longleftrightarrow \quad k(1 - b_i^{n+1}) < 1 - s$$

(3) Indeed, every time $f(a_v)$ dissapears, the difficulty decreases at least by ϵ, and this cannot happen infinitely many times since $d_{s,f}(Y^n, B^n) \geq 0$ by Lemma 1.4. □

Corollary 1.7. *Key Lemma holds if $s \neq 0$.*

Proof. Indeed, we can take $f(x) = 1 - x$ for $x < 1 - s$, $f(x) = 0$ for $x \geq 1 - s$ and $\epsilon = s$. □

Corollary 1.8. *Key Lemma holds if $m = \inf b_i^n > 0$.*

Proof. Indeed, for $s = 0$ we can take the function

$$f(x) = \begin{cases} 1 - x, & x \leq 1 - m \\ m, & 1 - m \leq x < 1 \\ 0, & x \geq 1 \end{cases}$$

and $\epsilon = m$. □

Corollary 1.9. *The previous two corollaries hold with* flipped *locus replaced by* flipping *locus.*

Proof. Change the function $f(x)$ at one point, so that $f(s) > 0$, resp. $f(m) > 0$. □

Corollary 1.10. *For any $\epsilon > 0$, the number of irreducible components C of*

$$\operatorname{Sing}\left(\cup_{b_i \geq \epsilon} B_i^n\right)$$

with $\operatorname{codim}_{Y^n} C = 2$ *and their multiplicities are bounded by a function of* (Y^0, B^0).

Proof. Take $s = \epsilon$ and a function $f(x)$ which satisfies a stronger inequality than (4b) if only $b_i > \epsilon$ are considered, namely:

$$f(1 - \sum m_i b_i) - \sum m_i f(1 - b_i) \geq (\sum m_i - 1)\epsilon$$

The proof of Lemma 1.4 shows that every codimension-2 component C with $\sum m_i > 1$ contributes at least $(\sum m_i - 1)\epsilon$ to $d_{\epsilon,f}$. Therefore, the number of such components and $(\operatorname{mult}_C (\cup_{b_i \geq \epsilon} B_i^n) - 1)$ are bounded by $d_{\epsilon,f}(Y^0, B^0)/\epsilon$. □

2. Termination of 4-dimensional klt flips

The following reduction argument is contained in [Kaw03]. Suppose

$$(X^0, D) \dashrightarrow (X^1, D) \dashrightarrow (X^2, D) \dashrightarrow \dots$$

is a sequence of klt flips for a fixed divisor $D = \sum d_j D_j$. Using only termination of $\mathbb{Q}$-factorial terminal flips (known by [Fuj04]), existence of $\mathbb{Q}$-factorial terminal flips (a recent work of Shokurov) and induction on the number $e(X, D)$ of discrepancies that are ≤ 0, one proves that this sequence can be covered by a sequence of flips as in Key Lemma so that for some $0 = \lambda(0) < \lambda(1) < \dots$ the pair $(Y^{\lambda(n)}, B^{\lambda(n)})$ is a crepant $\mathbb{Q}$-factorial terminalization of (X^n, D), possibly after truncating the sequence (cf. Backtracking Method and Crepant Descent in [K+92, Ch.6]; the method originates in [Kaw88]).

Further, one proves that after truncating the sequence the flipped locus does not contain any codimension-2 components of $\operatorname{Sing} X^n$.

If the flipped locus contains a codimension-2 subvariety C then the same is true for one of the covering flips upstairs, and the corresponding discrepancy is $a_v = 1 - s$, $s = \sum n_j d_j$ for some $n_j \in \mathbb{Z}_{\geq 0}$. Since there are only finitely many numbers s of this form in the interval $[0, 1)$, Key Lemma implies that after truncation the flipped locus no longer contains any components of codimension 2.

After this point, the flipping locus also cannot contain components of codimension 2 infinitely many times since then the rank of $H_{2(\dim X^n - 2)}(X^n)$ drops.

Finally, if $\dim X^n = 4$ then every flip is of type $(2, 1)$, $(1, 2)$ or $(2, 2)$, so either the flipping or the flipped locus must contain a codimension-2 subvariety. This proves the termination.

Since we were unable to prove Key Lemma in full generality, we only have the following corrolaries:

Corollary 2.1. *For $n \geq n_0$, the flipped and flipping loci of Y^n do not contain a codimension-2 subvariety $C \subset \operatorname{Supp} D$.*

Corollary 2.2. *Suppose that 0 is not an accumulation point from below of the discrepancies of klt pairs (X, D) with d_j in a fixed set and with a fixed number $e(X, D)$ of discrepancies that are ≤ 0. Then 4-dimensional klt flips terminate.*

References

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