THEOREMS ABOUT GOOD DIVISORS ON LOG FANO VARIETIES (CASE OF INDEX $r > n-2$)

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Introduction and formulating of the result

Definition 0-1. Let X be a normal complex variety of dimension n, $\Delta = \sum b_i E_i$ - a divisor with rational coefficients b_i such that $0 \le b_i < 1$, E_i - simple Weil divisors on X. Then X is said to have log- terminal (with respect to log-canonical divisor $K_X + \Delta$) singularities if the following conditions are satisfied:

(i) $K_X+\Delta$ is $\mathbb{Q}$-Cartier divisor, i.e. $N(K_X+\Delta) \in Div(X)$ for some natural N

(ii) There exists a resolution of singularities $f\colon Y \to X$ such that in the formula

$$K_Y+\tilde{\Delta} = f^*(K_X+\Delta) + \sum a_j F_j \qquad \text{for } a_j \in \mathbb{Q}$$

rational numbers a_j satisfy the condition $a_j > -1$. Here the support of the divisor $\tilde{\Delta}$ is strict transform of the divisor $\sum E_i$, F_j are simple exceptional divisors of the morphism f and $supp\ \tilde{\Delta} \cup \sum E_i$ is the divisor with only normal crossings.

Definition 0-2. Let X be a normal complex variety of dimension n. One says that X is a log Fano variety (with respect to log-canonical divisor $K_X+\Delta$) if the following conditions are satisfied:

(i) X has only log-terminal singularities with respect to $K_X+\Delta$

(ii) for some natural N Cartier divisor $-N(K_X+\Delta)$ is ample

In the case $n=2$ one usually calls about log Del Pezzo surfaces.

Definition 0-3. (i) Fano index of n-dimensional log Fano variety wi respect to $K_X+\Delta$ is the smallest positive rational number r_Δ such that $-(K_X+\Delta) = r_\Delta H$ in the group $DivX_{\mathbb{Q}}$ with the ample Cartier divisor H.

(ii) Fano spectrum is the set of rational numbers

$$FS_n = \left\{ r(X) \;\middle|\; \begin{array}{l} X \text{ is a log Fano variety of dimension } n \\ \text{with respect to canonical divisor } K_X \end{array} \right\}$$

(iii) the saturated Fano spectrum is the set

$$\underline{FS}_n = \left\{ r' \;\middle|\; \begin{array}{l} -K_X = r'H \text{ with the ample Cartier divisor } H, \\ r' \text{ is not nesessary minimal} \end{array} \right\}$$

Obviously, $\underline{FS}_n = \bigcup_{k=1}^{\infty} \frac{1}{k} FS_n$.

In [Sh] Shokurov proved the following theorem which is important for the classification of smooth Fano threefolds.

Theorem 0-4. Let X be a smooth Fano variety of dimension 3. Then in the linear system $|-K_X|$ there exists an irreducible smooth element.

Here we prove the following

Theorem 0-5. Let X be a log Fano variety of dimension n with respect to $K_X+\Delta$ with Fano index $r_\Delta > n-2$, $-(K_X+\Delta) = r_\Delta H$. Then

(i) in the linear system $|H|$ there exists an irreducible reduced element with only log-terminal singularities

(ii) the same is true for the linear system $|mH|$ for every natural number m.

In [OP] Shokurov proposed a number of interesting problems about FS_n , in particular

Conjecture 0-6. The set FS_n is upper semidiscontinious, i.e. for every x the set $FS_n \cap [x-\delta, x]$ is finite set for sufficiently small $\delta>0$.

It is easy to prove that F_n lies in $]0, n+1]$ and $r=n+1$ iff X is $\mathbb{P}^n$, $r=n$ iff X is quadric.

In [F1] T.Fujita described the set $F_n \cap]n-1, n]$ and corresponding Fano varieties. He showed that all these varieties have Δ-genera zero, so it follows from [F2] that they are either cones over rational normal curves C_d in $\mathbb{P}^d$ $(r=n-1+\frac{2}{d})$ or cones over Veronese surface S_4 in $\mathbb{P}^5$ $(r=n-\frac{1}{2})$. So, conjecture 0-6 is true for $FS_n \cap [n-1, n+1]$.

In [A] the author proved

Theorem 0-7. FS_2 is upper semidiscontinious, moreover one has only the following limit points: 0 and $1/k$ for every natural k.

From 0-5(i) and 0-7 we have

Corollary 0-8. For $n>2$ $FS_n = \underline{FS}_2 + (n-2)$

Therefore, the conjecture 0-6 is true for the set $FS_n \cap [n-2, n-1]$. Moreover, one has only the following limit points: $n-2$ and $n-2+\frac{1}{k}$ for every natural k.

Proof of the corollary. Let $-K_X = rH$ and $r>n-2$. Then a general element $X_{n-1} \in |H|$ is a log Fano variety too and

$$-K_{X_{n-1}} = (r-1)H\big|_{X_{n-1}} \qquad \text{and } r-1 > (n-1)-2.$$

Repeating this process $(n-2)$-times we obtain a log Del Pezzo surface $X_2 \in |H|^{n-2}$ and $-K_{X_2} = (r-n+2)H\big|_{X_2}$, so $r-n+2 \in \underline{FS}_2$. On the contrary, if we have the log Del Pezzo surface Y and $-K_Y = r'H$ then $(n-2)$-multiple generalized cone over X_2 (see construction 0-9 below) is a log Fano variety of dimension n and of Fano index $r = r' + (n-2)$. ∎

The following construction is due to T.Fujita, [F1].

Construction 0-9. Let X be a log Fano variety dimension n and $-K_X = r'H$. Let us consider the line bundle $\mathcal{O} \oplus \mathcal{O}(-H)$ on X and let $Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-H))$. Let P be the negative section of Y, $P{=}X$. It is easy to prove that P is contractible to a point and we obtain the morphism $f\colon Y \to X'$ with $\mathbb{Q}$-Gorenstein variety X' and $K_Y = f^*K_{X'} + (r-1)P$. Therefore, X' is a log Fano variety of dimension $n+1$ and with Fano index $r = r'+1$. This variety is called by a generalized cone over variety X.

Below we assume the general case (i.e. Δ is arbitrary) and denote by r the number $\frac{-K_X \cdot H^{n-1}}{H^n}$. Note that $r = r_\Delta + \frac{\Delta \cdot H^{n-1}}{H^n} \geq r_\Delta > n-2$.

1. Proof of the theorem 0-5(i)

With the same assumptions as above one has

Proposition 1-1. $h^0(H) > 0$.

Proof. For $x \geq -(n-2)$ the divisor $-(K_X+\Delta) + rH$ is an ample $\mathbb{Q}$-divisor, so it follows by standard arguments from Kawamata-Fiehweg vanishing theorem (see f.e. [KMM]) that $h^i(xH) = 0$ for $i>0$, $x \geq -(n-2)$, $\chi(xH) = h^0(xH)$. If $h^0(H) = 0$ then the polynomial $\chi(xH)$ has the zeros $-1, -2 \ldots -(n-2), 1$. Besides $\chi(0\cdot H) = 1$ and $\chi(xH)$ has the main coefficient $\frac{d}{n!}$, where $d = H^n$.

Therefore

$$\chi(xH) = \frac{1}{n!}\,(x+1)\ \ldots\ (x+(n-2))\ (x-1)\ (dx-n(n-1)) =$$
$$= \frac{1}{n!}\,(dx^n + [n(n-3)\tfrac{d}{2} - n(n-1)]\ x^{n-1} + \ldots)$$

On the other hand, by Riemann-Roch

$$\chi(xH) = \frac{1}{n!}\,(xH)^n + \frac{1}{2(n-1)!}\,(-K_X)\ (xH)^{n-1} + \ldots =$$

$$= \frac{1}{n!}\ (dx^n + \frac{1}{2}\ nrdx^{n-1} + \dots)$$

So, we have $r = n - 3 - \frac{2(n\ 1)}{d}$. But this contradicts to the condition $r > n-2$. It is not difficult to write the polynomial $\chi(xH)$ precisely

$$\chi(xH) = \frac{1}{n!}\ (x+1)\ \dots\ (x+(n-2))\ \cdot$$

$$\cdot\ (dx^2 + \frac{1}{2}\ d(nr - (n-2)(n-1))x + (n(n-1))$$

In particular, $h^0(H) = \frac{1}{2}\ d(r-n+3) + n - 1$.

Below we use Kawamata's techniques as it described in [R].

<u>Construction</u> <u>1-2</u>. There is a resolution of singularities $f\colon Y \to X$ a a divisor with normal crossing $\sum F_j$ and constants a_j , r_j , p_j and q such that

(1) $K_Y+\tilde{\Delta} = f^*(K_X+\Delta) + \sum a_jF_j$, $\tilde{\Delta} = \sum b_jF_j$ where $a_j > -1$ and a_j is not equal to zero only if F_j is exceptional for f.

(2) $f^*|H| = |L| + \sum r_jF_j$ with free linear system $|L|$, $r_j \in \mathbb{Z}$ and $r_j \geq 0$

(3) $qf^*H - \sum p_jF_j$ - ample $\mathbb{Q}$-divisor where p_j , $q \in \mathbb{Q}$ and $0<p_j$, $q \ll 1$.

Consider constants $c \in \mathbb{Q}$, $c \geq 0$, $b \in \mathbb{Z}$ and the divisor

$$N = N(b,c) = bf^*H + \sum(-cr_j + a_j - p_j)F_j - (K_Y+\tilde{\Delta}) =$$
$$= cL + f^*(b-c+r_\Delta)H - \sum p_jF_j$$

This divisor is ample on Y if $b-c+r_\Delta > const > 0$ and its fractional part is supported in $\sum F_j$. Let $c=\min_J(a_j+1-p_j-b_j)/r_j$ (J is the set of index with $r_j \neq 0$). Changing p_j we can assume that minimum is achieved only for one index $j=0$. Then $-cr_0 + a_0 - p_0 - b_0 = -1$ and $\sum \ulcorner (-cr_j + a_j - p_j)F_j - \tilde{\Delta} \urcorner = A - B$ ($\ulcorner\ \urcorner$ means upper integer part) where $B = F_0$, A consists of components F_j exceptional for f. Then $H^0(Y,bf^*H+A) \twoheadrightarrow H^0(B,bH'+A')$ and $H^0(B,H'+A') = 0$ where $H'= f^*H|_B$ and $A'= A|_B$.

Besides, $H^0(bH'+ A') = \chi(bH'+ A')$.

<u>Proposition</u> <u>1-3</u>. For all j, $r_j < a_j + 1$.

<u>Proof</u>. Let us assume the opposite. Then $c = \min(a_j+1-p_j-b_j)/r_j < 1 - const$. Consequently for $b \geq (n-3)$ we have $b+c-r > const > 0$, since $r>n-2$. Consider the polynomial of degree $n-1$ $\chi(xH'+A') = h^0(xH'+ A')$ for $x \geq -(n-3)$. This polynomial has the zeros -1, -2 ... $-(n-3)$ and if the set J is not empty (i.e. $Bas|H| \neq \emptyset$) then it has also the zero in the point 1 by the construction 1-2. Besides, $\chi(A') = 1$ since the divisor A' is effective and $h^0(A) = 1$.

Consider two cases.

Case 1. $H'^{n-1} = 0$, i.e. $f_*B = 0$. Then

$$\chi(xH' + A') = -\frac{1}{(n-3)!}(x+1)\dots(x+n-3)(x-1)$$

But $\chi(xH' + A') = h^0(xH' + A') > 0$ for $x \gg 0$ and we obtain a contradiction.

Case 2. $d' = H'^{n-1} \neq 0$, i.e. $f_*B = B_1$ is a base component of the linear system $|H|$. Then

$$\chi(xH' + A') = \frac{1}{(n-1)!}(x+1)\dots(x+n-3)(x-1)(d'x-(n-1)(n-2)) =$$

$$= \frac{1}{(n-1)!}(d'x^{n-1} + (n-1)[\tfrac{n-4}{2}d' - n + 2]x^{n-2} + \dots)$$

On the other hand by Riemann-Roch

$$\chi(xH' + A') = \frac{1}{(n-1)!}(xH' + A')^{n-1}$$

$$- \frac{1}{2(n-2)!}K_B(xH' + A')^{n-2} + \dots =$$

$$= \frac{1}{(n-1)!}(d'x^{n-1} + \tfrac{1}{2}(n-1)(A' - \tfrac{1}{2}K_B)\, H'^{n-2}\, x^{n-2} + \dots)$$

Consequently, $(n-4)d' - (n-2) = (2A' - K_B)H'^{n-2}$ (*)

Estimate the right part of this inequality. Firstly, $A'H'^{n-2} \geq 0$ since A' is effective and H' is numerically effective. Now prove that $-K_B \cdot H'^{n-2} \geq (-K_X - B_1)\cdot B_1 \cdot H^{n-2}$. It is sufficient to consider only two-dimensional case. Indeed, we have only to restrict B_1 (and B) on a general surface S_1 (S) from the linear system $|mH|^{n-2}$ ($|f^*mH|^{n-2}$) for sufficiently large m.

Thus, S_1 is a normal surface, $f\colon S \to S_1$ is some resolution of singularities, B and B_1 are curves on S and S_1 respectively. The morphism f splits into decomposition $f = \pi^0 g$ where $g\colon S \to T$ and $\pi\colon T \to S_1$, π is the minimal desingularization, $C = g(C)$ is Gorenstein curve on T, probably singular. Then firstly (in the numerical notation)

$$-K_B = -K_{B_1} - M \geq -K_C$$

where $M \geq 0$ is the degree of the normalization.

Secondly, $-K_C = (-K_T - C)C$, $C = \pi^* B_i - \sum \tau_i B_i$, $\tau_i \geq 0$. Here E_i are exceptional divisors of the resolution π. We have:

$$-C^2 = -B_1^2 - (\sum \gamma_i E_i)^2 \geq -B_1^2$$

since the quadratic form of intersection $(E_i \cdot E_j)$ is negatively defined.

$$-K_T \cdot C = -K_B \cdot B_1 + \sum \gamma_i \cdot K_T \cdot E_i \geq -K_B \cdot B_1$$

since $K_T \cdot E_i = 2p_a(E_i) - 2 - E_i^2 \geq 0$ since the resolution π is minimal.

So we proved that

$$-K_B \cdot H'^{n-2} \geq (-K_X - B_1) \cdot B_1 \cdot H^{n-2} = rd' - B_1^2 \cdot H^{n-2}$$

Recall that B_1 is a base component of the linear system $|H|$. So

$$|H| = k\, B_1 + C \quad \text{and } B_1 \cdot C \cdot H^{n-2} \geq 0$$

Consequently $B_1^2 \cdot H^{n-2} \leq B_1 \cdot H^{n-1} = d'$.

Now let us return to the equality (*). We showed that the right part is not less than $(r-1)d' > (n-3)d'$ and the left one is less than $(n-4)d'$. We obtain a contradiction and proof of the proposition 1-3 is finished. ∎

1-4. Proof of the theorem 0-5(i).

Consider the linear system $|H|$. It is not empty by the proposition 1-1. Firstly it has no base components. Otherwise, for resolution we should have a divisor F_j with $a_j = 0$ and $r_j \geq 1$ that contradicts to the proposition 1-3. Secondly for a general divisor $X_{n-1} \in |H|$ one has dim Sing $X_{n-1} < n-2$. Otherwise one can easy prove that there exists F_j with $a_j \leq 0$ and $r_j \geq 1$.

Now from the connectedness theorem ([R], lemma 0-9(iii)) it follows that general divisor is irreducible. Now general element $X_{n-1} \in |H|$ is hypersurface in a normal variety, nonsingular in codimension 1, consequently it is a normal variety.

The morphism $f\colon Y \to X$ gives a desingularization for X_{n-1}, one has $f_{n-1} = f|_{Y_{n-1}} : Y_{n-1} \to X_{n-1}$ and $Y_{n-1} \in |Y|$. It is easy to verify that

$$K_{Y_{n-1}} + \tilde{\Delta}|_{Y_{n-1}} = f^*(K_{X_{n-1}} + \Delta) + \sum(a_j - r_j)\, F_j|_{Y_{n-1}}$$

By the proposition 1-3 $a_j - r_j > -1$ and we are done.

In the extremal case $n=2$ we have to refine our arguments because some formulars above loose the sence. Nevertheless these arguments work and much more strong theorem is true

Theorem 1-5. Let X be a log Del Pezzo surface with respect to $K_X+\Delta$ and D be an arbitrary numerically effective Cartier divisor. Then $|D| \neq \emptyset$ and the linear system $|D|$ contains a nonsingular element. (Note that for dimension 1 "log-terminal" means nonsingular).

Proof. The proof of the proposition 1-1 goes without any difficulties. The respective equality is $-D^2 - 2 = -K_X \cdot D \geq 0$ and we obtain a contradiction. In the proof of proposition 1-3 we have $\chi(xH' + A') = h^0(xH' + A')$ for $x \geq 1$ because $(x-c)D - K_X$ is ample for $x \geq 1$. Therefore we have $\chi(xH' + A') = 0$.

In the case $D' \equiv 0$ we have $\chi(xH'+A') \equiv 0$ but it contradicts to $\chi(xH'+A') = h^0(xH'+A') > 0$ for x sufficiently large and divisible.

In the case $D' \not\equiv 0$ $\chi(xH'+A') = d'(x-1)$ and we have the equality

$$-2d' = 2A' - K_B \quad (*)$$

and $-K_B \geq (-K_B - B_1)\cdot B_1 \geq -d'$, so we obtain a contradiction again. Therefore we have the proposition, corresponding to 1-3.

Finally item 1-4. We prove analogously that the linear system $|D|$ has no base components and a general element is reducible. If $D^2 > 0$, the end of proof is the same. If $D^2 = 0$, then $|D|$ is a pencil without any base points and a general elements is again nonsingular.

■

2. The theorems for multiple |mH|

Lemma 2-1. Let C be a nonsingular curve of the genus $g>0$ and $|D|$ is a complete linear system on C of degree d. Then

(i) for $d \geq 2g-1$ $|D| \neq \emptyset$

(ii) for $d \geq 2g$ $|D|$ is free

(iii) for $d \geq 2g+1$ $|D|$ is very ample.

Proof. By Riemann-Roch.

Lemma 2-2. Let $C \in |H|^{n-1}$ is a nonsingular curve, existing by the theorem 0-5(*i*). Then $H^0(X, mH) \twoheadrightarrow H^0(C, mH)$ for $m \geq 1$.

Proof. By induction, using the fact that $h^1(X_i, (m-1)H) = 0$ for $X_i \in |H|^{n-1}$, $i \leq n-2$ (by vanishing theorems).

Proposition 2-3. In the same notation

(i) base locus $Bas|H|$ is a finite set of points

(ii) if we denote by

$$t = (-K_X - (n-2)H)H^{n-1} \geq (-K_X-\Delta-(n-2)H)H^{n-1} > 0$$

then for $t \geq 2$ or $m \geq 2$ one has $Bas|mH| \neq \emptyset$

(iii) for $t \geq 3$ or $m \geq 3$ $|mH|$ is very ample.

(iv) for $m \geq 2$ a general element of $|mH|$ has only log-terminal singularities with respect to $K_{mH}+\Delta|_{mH}$

Proof. (i), (ii), (iii) follow immediately from the lemmas 2-1 and 2-2 since $d = mH^n$ and $2g-2 = (K_X + (n-1)H)H^{n-1}$. (iv) follows from (ii).

3. The case r = n-2

Proposition 3-1. If $-K_X$ is linearly equivalent to $(n-2)H$ then

$$\chi(xH) = \frac{1}{n!}(x+1)\ldots(x+n-3)(dx^3 + \frac{3}{2}d(n-2)x^2 +$$

$$+ [\ 2n(n-1) + \frac{1}{2}\ d(n-2)^2\ ]\ x + n(n-1)(n-2)\ ),\ d = H^n,$$

otherwise we have preceding formula, see 1.1. In particular,

$$h^0(H) = d^n/2 + n \text{ or } d^n/2 + n - 1,\ h^0(H) > 0.$$

Proof is analogous to that of 1-1, but instead of $\chi(-(n-2)H) = 0$ we have $\chi(-(n-2)H) = \chi(K_X) = (-1)^n$, if $-K_X \sim (n-2)H$.

Proposition 3-2. For corresponding constants one has $r_j \le a_j + 1$

Proof is analogous to that of 1-3.

Corollary 3-3. A general element of the linear system $|H|$ is reduced and has only simple quadratic singularities in codimension 1.

Proof. As in [R].

Remark 3-4. It would be nice to prove the proposition 1-3 (with strong inequalities) for the case $r=n-2$ too. Unfortunately, we loose in this case one more zero of the polynomial $\chi(xH)$ and we don't know how to compensate this. The proof of Shokurov's theorem [Sh] uses some results about classification of surfaces and it is difficult to generalize them.

Note that the strong analog of Shokurov's theorem (i.e. for smooth Fano variety and smooth divisor) follows immediately from mentioned strong inequalities. Assuming the latter Mukai in [Mu] gave a classification of Fano manifolds with $r=n-2$ continuing results of Iskovskich and Mori-Mukai from dimension 3 to higher dimensions.

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