On quotient surfaces of P^2 by a finite group

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Let X be a normal algebraic surface over the field **C**. The canonical Weil divisor K_X is defined on X. The surface X is called a *del Pezzo log surface* if it has only quotient singularities, and some multiplicity $-nK_X$ is an ample Cartier divisor. All these surfaces are rational. Quotient surfaces of P^2 by finite groups are examples of del Pezzo log surfaces with Pic $X = \mathbf{Z}$. We show that these do not exhaust the class of such surfaces. It is known that there are three surfaces with ample anti-canonical divisor and with Pic $X = \mathbf{Z}$, each of which has exactly one singular point: E_6, E_7, and E_8, respectively (see, for example, [1], [2]). It turns out that they do not have the form P^2/G.

Notation. The symbols G and A denote finite subgroups of PGL(3, **C**), and the symbols G' and A' denote finite subgroups of GL(3, **C**) such that $\alpha(G') = G$ and $\alpha(A') = A$ under the natural homomorphism $\alpha : \mathrm{GL}(3, \mathbf{C}) \to \mathrm{PGL}(3, \mathbf{C})$.

We recall some facts about two-dimensional quotient singularities. They are all rational. For the minimal resolution there is attached a tree of rational curves that form a weighted graph of type A_n, D_n, or E_n if a vertex with weight $-C_i^2$ is associated with each rational curve C_i, and two vertices are joined by an edge when $C_i \cdot C_j = 1$. All such graphs are listed in [3]. Graphs with weight 2 at all vertices correspond to Du Val singularities. These singularities are denoted by A_n, D_n, and E_n. The toroidal singularities are precisely those having weighted graphs of type A_n. Another description of them is: quotients by Abelian groups.

Lemma. *Let (U, P) be a toroidal singularity, and $i : U \to U$ an involution leaving the point P fixed. Then the quotient singularity (V, Q), $V = U/i$, has graph of type A_n or D_n.*

The proof of the lemma is immediate.

Theorem. *Among the quotient surfaces P^2/G there are no surfaces with precisely one singular point E_6, E_7, or E_8, respectively.*

Proof. Let $G' \subset \mathrm{GL}(3, \mathbf{C})$ be a finite group acting on $V \cong \mathbf{C}^3$. Three cases are possible: 1) G' is an imprimitive group, that is, there is a decomposition $V = V_1 \oplus V_2 \oplus V_3$ such that dim $V_i = 1$ and $g(V_i) = V_j$ for any element g of G'. In this case G' contains a normal Abelian subgroup $A' = \{a \in G' \mid a(V_i) = V_i,\ i = 1, 2, 3\}$ and $G'/A' \subset S_3$. 2) G' is a reducible group, that is, there is a decomposition $V = V_1 \oplus V_2$ such that dim $V_1 = 1$, dim $V_2 = 2$, and $G(V_i) = V_i$, $i = 1, 2$. Since we are interested only in the image of G' in PGL(3, **C**), it can be assumed that $G' = 1 \oplus H$, $H \subset \mathrm{GL}(2, \mathbf{C})$. 3) G' is a primitive group, that is, it does not occur in the preceding cases.

These cases will be analyzed successively.

1) We have that $P^2/G = (P^2/A)/(G/A) = (P^2/A)/(G'/A')$. The surface P^2/A has three or fewer singular points, which correspond to the spaces V_1, V_2, and V_3. These singularities are toroidal. The group G'/A' permutes them in some way. With the help of the lemma we see that there are no singularities E_6, E_7, nor E_8 on P^2/G.

2) It can be assumed that H is primitive, for otherwise G' falls in the first case. All such groups are listed in [4], Lemmas 2.3–2.5. We use the notation from that paper. Let us consider the group $\mathfrak{G}_1$ with $Z(\mathfrak{G}_1) = \{\pm E\}$ in Lemmas 2.3, 2.4, and 2.5, or the group $\mathfrak{G}_2$ in Lemma 2.5. Let x, y, z be homogeneous coordinates on P^2. At infinity (that is, on the hyperplane $x = 0$) there are three types of points with non-trivial isotropy subgroup. The isotropy subgroup of the point (0, 0, 1) has the form $x \to x$, $y \to iy$, $z \to -iz$. This point is mapped into a non-singular point on P^2/G if the roots of degree $4k$ are added, and into the singularities A_1 otherwise. But if the 4th-degree roots are added to the groups under consideration, then we get groups generated by reflections, therefore, we have a toroidal singularity on the finite part after addition of the roots of degree $4k$. The "infinite points" always pass into toroidal singularities.

We now consider the group $\mathfrak{G}_2$ in Lemma 2.3. The element $1 \oplus (-\alpha A)$ has the form $x \to x$, $y \to -\alpha\varepsilon y$, $z \to -\alpha\varepsilon^2 z$, where $\varepsilon = e^{2\pi i/3}$, in a suitable basis. This element together with the element $x \to x$, $y \to \mu y$, $z \to \mu z$ generates the isotropy subgroup of the points (0, 0, 1) and (0, 1, 0). For these points to pass into non-singular points on P^2/G it is necessary that $3^r | d$; but then $\mathfrak{G}_2$ coincides with the group $\mathfrak{G}_1$ already analyzed.

Finally, the last case concerns the group $\mathcal{G}_2$ in Lemma 2.4. Let

$$C=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -i\\ i & -1\end{pmatrix}.$$

In a suitable basis the element $1 \oplus \beta C$ has the form $x \to x, y \to \eta\beta y, z \to \eta^{-1}\beta z$, where $\eta = e^{\pi i/4}$. This element together with the elements $x \to x$, $y \to \beta^2 y$, $z \to \beta^2 z$ and $x \to x$, $y \to \nu y$, $z \to \nu z$ generates the isotropy subgroup of the point (0, 0, 1). If $r \neq 3$, then, again, for this point to pass into a nonsingular point it is necessary that $2^r | d$, and once more we arrive at the case of the group $\mathcal{G}_1$. Let $r = 3$. Exactly as in the case of the group $\mathcal{G}_2$ we see from Lemma 2.3 that $3|d$. By formulae in [4], it can be verified that $\mathcal{G}_2$ is generated by reflections for $d = 3$. Consequently, there is a toroidal singularity on the finite part when the roots of degree $3k$ are added.

3) All three-dimensional primitive groups are listed to within scalar matrices in [5]; there are six in all. It can be verified directly that none of the desired surfaces occurs in this case. The theorem is proved.

References

[1] P. Du Val, On isolated singularities of surfaces which do not affect the condition of adjunction. I, II, III, Proc. Cambridge Phil. Soc. **30** (1934), 453-465, 483-491. Zbl. **10**-176, **10**-177.

[2] D. Bindschadler, L. Brenton, and O. Drucker, Rational mappings of del Pezzo surfaces, and singular compactifications of 2-dimensional affine varieties, Tôhoku Math. J. **36** (1984), 591-609. MR **86i**:14011.

[3] E. Brieskorn, Rationale Singularitäten komplexer Flächen, Invent. Math. 4 (1968), 336-358. MR **36** # 5136.

[4] W.C. Huffman, Polynomial invariants of finite linear groups of degree two, Canad. J. Math. **32** (1980), 317-330. MR **81g**:15033.

[5] H.F. Blichfeldt, Finite collineation groups, Chicago Univ. Press, Chicago 1917.

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