3. CLASSIFICATION OF LOG CANONICAL SURFACE SINGULARITIES: ARITHMETICAL PROOF

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(3.0.0). Notation. Let (X, P) be a germ of a normal surface singularity and $B = \sum b_i B_i$ a formal sum of irreducible Weil divisors, passing through P, with rational coefficients $0 \leq b_i \leq 1$. Since X is normal, we can assume that P is the only singularity of X. Also, we have a well defined linear equivalence class of canonical Weil divisors K_X.

We use the usual definitions for *log canonical, log terminal and purely log terminal* (2.13).

(3.0.1). If $B = \emptyset$ and the characteristic of the base field is 0, log terminal singularities of surfaces are the same as quotient singularities [Kawamata84] and were classified by [Brieskorn68]. [Iliev86] contains an arithmetical proof.

In the case B is *reduced*, i.e. all the $b_i = 1$, [Kawamata88] classified all log canonical and log terminal singularities (the latter turn out to be also purely log terminal with one trivial exception: when X is nonsingular and B consists of two normally crossing nonsingular curves). This classification is given in Fig.3. The notation is explained in (3.1).

The proof of [Kawamata88] is slightly tricky and uses the log canonical cover of (X, P). Arithmetical proofs were given in [Sakai87] for the case $b_i = 0$ and by S. Nakamura in an appendix to [Kobayashi90].

(3.0.2). Here we suggest a purely arithmetical and quite elementary approach for the classification. The idea is the following: let $f : Y \to X$ be the minimal resolution of the singularity (X, P) (a priori not a good resolution of (X, P)).

Let $f_*^{-1}C \subset Y$ denote the birational transform of a curve $C \subset X$. Write

$$K_Y + \sum f_*^{-1} B_i + \sum E_j = f^*(K_X + \sum B_i) + \sum a_j E_j.$$

Then for any $j = 1, \ldots, n$, by the adjunction formula, we have

$$\begin{aligned} 2p_a(E_j) &= E_j(K_Y + E_j) + 2 = \\ &= E_j(f^*(K_X + B) + \sum_k a_k E_k - \sum f_*^{-1} B_i - \sum_{k \neq j} E_k) + 2 = \\ &= E_j(\sum a_k E_k - \sum f_*^{-1} B_i - \sum_{k \neq j} E_k) + 2 \end{aligned}$$

Therefore we get the following system of n linear equations in n variables

$$(*) \qquad \sum_{k=1}^{n} a_k E_k \cdot E_j = -c_j,$$

where $c_j = 2 - 2p_a(E_j) - (\sum f_*^{-1} B_i + \sum_{k \neq j} E_k) E_j$. Equivalently,

$$(**) \qquad \sum_{k=1}^{n} (a_k - 1) E_k \cdot E_j = -d_j,$$

where $d_j = 2 - 2p_a(E_j) + E_j^2 - \sum f_*^{-1} B_i \cdot E_j$.

(3.0.3). Now our strategy is very simple: solve the system $(*)$, find the a_k and check the conditions $a_k \geq 0$.

(3.0.4). Some of the formulas for the coefficients a_k are contained in [Alexeev89, 4.7,4.8]. Note also that in the log terminal case with $B = \emptyset$, our treatment has some intersections with [Iliev86]. However, our proof is more explicit and direct.

J. Kollár points out that the present proof works in any characteristic. This follows from the fact that the system $(*)$ has a unique solution independent of the characteristic of the base field.

3.1. Solution of $(*)$.

(3.1.0). First, note that $(*)$ does have a unique solution since by [Mumford61] the matrix $(E_k \cdot E_j)$ is negative definite.

(3.1.1). The *weighted dual graph* Γ of the resolution $f : Y \to X$ is the following: each curve E_j corresponds to a vertex v_j. Two vertices v_{j_1} and v_{j_2} are connected by an edge of weight m if the corresponding curves intersect: $E_{j_1} \cdot E_{j_2} = m$. Each vertex v_j has a positive weight $n_j = -E_j^2$.

Since the resolution f is minimal, we have $d_j = 2 - 2p_a(E_j) + E_j^2 - \sum f_*^{-1} B_i E_j \leq 0$ for all j.

(3.1.2). By (2.19.3) every coefficient of the inverse matrix of $(E_k \cdot E_j)$ is strictly negative. Therefore, $(**)$ implies that either all $d_j = 0$, and then for all k, $a_k - 1 = 0$ or at least one $d_j < 0$, and for all k, $a_k - 1 < 0$. The former happens only if all $p_a(E_j) = 0$, $E_j^2 = -2$ and $E_j \cdot \sum f_*^{-1} B_i = 0$. Such singularities (and the corresponding graphs) are called *Du Val singularities* (resp. *Du Val graphs*).

The following result is easy.

(3.1.3). Lemma. *(cf. [Alexeev89,3.2(ii-iii)]) Let Γ be a weighted graph corresponding to a minimal resolution, in particular such that all $d_j \leq 0$. Let $\Gamma' \subset \Gamma$, $\Gamma' \neq \Gamma$ be a subgraph in the sense that all the vertices of Γ' are at the same time vertices of Γ with the same weight n_j, the weights of edges of Γ' and p_a of vertices in Γ' do not exceed the corresponding weights and p_a in Γ, and $E_j \cdot \sum f_*^{-1} B_i$ in Γ' do not exceed the corresponding $E_j \cdot \sum f_*^{-1} B_i$ in Γ.*

Then the corresponding coefficients satisfy $a_k \leq a'_k$ and if Γ is not a Du Val graph, then $a_k < a'_k$.

Proof. Compare the corresponding systems $(**)$ of linear equations and use (3.1.2). □

(3.1.4). Suppose that $\Gamma' = \{v_1\}$ and $p_a(E_1) = 1$. Then in $(*)$ $c_1 = 2 - 2p_a(E_1) - 0 = 0$ and $a'_1 = 0$. If E_1 is a smooth elliptic curve, this is Case 4 of Fig.3. If E_1 is a rational curve with a node then after a single blow up we are in Case 5 of Fig.3. If E_1 is a rational curve with a cusp it is easy to show that after two blow ups one gets a log discrepancy $a_3 = -1$, so this is not a log canonical singularity.

(3.1.5). Suppose that $\Gamma' = \{v_1, v_2, \ldots, v_l\}$ is a circle of smooth rational curves. Then in $(*)$ $c_j = 2 - 0 - 2 = 0$ and all $a'_j = 0$. This is Case 5 of Figure 3. Note that all the curves E_j should intersect normally: if a circle contains two or three vertices and two corresponding curves have a common tangent, or three curves intersect at one point, then two or one blow ups give a log discrepancy $a'_3 = -1$.

(3.1.6). Now (3.1.2-5) imply that:

(3.1.6.1). The graph of a log canonical singularity does not contain a vertex v_j with $p_a(E_j) > 1$ or an edge of weight > 2.

(3.1.6.2). If $\Gamma \neq \Gamma'$ as in (3.1.4) or (3.1.5), then Γ contains only vertices that correspond to smooth rational curves, all edges are simple, i.e. of weight 1, and Γ is a tree.

From now on we always assume that we are in this final case.

(3.1.7). For any subgraph $\Gamma' \subset \Gamma$, we define $\Delta' = \Delta(\Gamma')$ as the absolute value of the determinant of the submatrix $(E_k \cdot E_j)$, made up by the columns and rows corresponding to the vertices of Γ'.

Note that if Γ' is a disjoint union of graphs Γ_1 and Γ_2, then $\Delta' = \Delta_1 \cdot \Delta_2$. We set $\Delta(\emptyset) = 1$ by definition.

The following lemmas are easy exercises.

3.1.8 Lemma. *Let Γ be a graph with simple edges, v a vertex of Γ of weight n, and $v_1, \dots, v_s$ the vertices adjacent to v. Then*

$$\Delta(\Gamma) = n \cdot \Delta(\Gamma - v) - \sum_i \Delta(\Gamma - v - v_i).$$

3.1.9 Lemma. *Let Γ be a tree with simple edges, v_{j_1}, v_{j_2} two vertices of Γ. Then the (j_1, j_2) cofactor of the matrix $(E_k \cdot E_j)$ is*

$$A_{j_1 j_2} = (-1)^{j_1 + j_2} M_{j_1 j_2} = -(-1)^n \Delta(\Gamma - (\text{path from } v_{j_1} \text{ to } v_{j_2}))$$

Note that since Γ is a tree there is a unique (shortest) path joining v_{j_1} and v_{j_2}.

(3.1.10). The previous lemma gives the solution of $(*)$:

$$(***) \qquad \begin{aligned} a_j &= \frac{1}{\Delta(\Gamma)} \sum_{k=1}^{n} \Delta(\Gamma - (\text{path from } v_j \text{ to } v_k)) \cdot c_k, \\ c_k &= 2 - (\sum f_*^{-1} B_i + \sum_{l \neq k} E_l) E_k. \end{aligned}$$

Here $(\sum f_*^{-1} B_i + \sum_{l \neq k} E_l) E_k$ is the number of connections of the vertex v_k with adjacent vertices (among $\sum f_*^{-1} B_i$ and the other E_l). Therefore, $c_k = 0$ if and only if v_k has exactly 2 neighbours, $c_k = 1$ if it has 1 neighbour and $c_k < 0$ if if has ≥ 3 neighbours. By $(***)$, a_j is a sum of c_k with positive coefficients. We are interested in the cases when $a_j \geq 0$, therefore we call vertices with $c_k = 1$ (resp. $c_k < 0$) *bonus* (resp. *penalty*) vertices.

Now our aim is to simplify the use of the formulas $(***)$.

(3.1.11). We need the following well known description of weighted chains. Every weighted chain with positive integer weights (from the left to right) $n_1, \dots, n_s \geq 2$ corresponds in unique way to the pair (Δ, q), where $\Delta = \Delta(\Gamma)$ and $1 \leq q < \Delta$ is an integer coprime to Δ defined by:

$$\frac{\Delta}{q} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\cdots \cfrac{1}{n_s}}}$$

Let us show how to get this description. Let v be the end vertex of the chain Γ. Then by (3.1.8), $\Delta = \Delta(\Gamma)$ can be expressed in terms of $q = \Delta(\Gamma - v)$ and $\Delta(\Gamma - v - v_1)$, then $\Delta(\Gamma - v)$ can be expressed in terms of $\Delta(\Gamma - v - v_1)$ and $\Delta(\Gamma - v - v_1 - v_2)$ and so on, the last determinant will be $\Delta(\emptyset) = 1$. One can easily see that this procedure is nothing other than the Euclidean algorithm for finding the greatest common divisor, so $(\Delta, q) = 1$, and one gets the given formula.

3.1.12 Lemma. *Suppose that a graph Γ contains a subgraph Γ' such that Γ' is a chain with weights $n_j \geq 2$ and the interior vertices of this chain have no other neighbors in Γ or $\sum B_j$. Let v_{j_1} be one of the middle vertices, a_{j_1} the corresponding log discrepancy of Γ. Then the graph of the function a_j at the vertex v_{j_1} is concave up if $a_{j_1} \geq 0$ and is concave down if $a_{j_1} \leq 0$.*

Proof. Note that from $(*)$

$$a_{j_1 - 1} - n_{j_1} a_{j_1} + a_{j_1 + 1} = 0,$$

so that

$$|a_{j_1}| = \left|\frac{a_{j_1 - 1} + a_{j_1 + 1}}{n_{j_1}}\right| \leq \left|\frac{a_{j_1 - 1} + a_{j_1 + 1}}{2}\right|.$$

The rest is obvious. ☐

3.1.13 Lemma. *Let Γ be a tree with simple edges and all weights $n_j \geq 2$ (all these conditions hold in our situation). Then all the log discrepancies of Γ are nonnegative (resp. positive) if and only if the same holds for all vertices with at least 3 neighbours and for all vertices neighbouring $\sum f_*^{-1} B_i$.*

Proof. Indeed, if $\Gamma' \subset \Gamma$ is a subchain such that each middle vertex has exactly 2 neighbours and one of this middle vertices has $a_{j_1} \leq 0$ (resp. $a_j < 0$), then by (3.1.12) the same holds for the ends of Γ'.

Moreover, we can exclude the vertices with exactly 1 neighbour, because from $(*)$ we have

$$a_{j_1+1} - n_{j_1} a_{j_1} = -1$$

and $a_{j_1} \le 0$ implies $a_{j_1+1} < a_{j_1}$. □

(3.1.14). We explain the notation of Fig.3. We consider a minimal resolution $f : Y \to X$ (with the exception of Case 5). ∘ denotes an exceptional curve of f, • denotes (local branches of) B_i. Long empty ovals denote any chain (Δ, q), attached at an end.

3.2. The case $B = \emptyset$. We first consider several simple possibilities for the graph Γ

(3.2.1). Let Γ be a chain. Then by (3.1.13) Γ corresponds to a log terminal singularity, because none of the vertices has ≥ 3 neighbours.

(3.1.10) gives the formula for the log discrepancies. Let v_j be a vertex of Γ, so that $\Gamma - v_j = \Gamma_1 - \Gamma_2$ is a disjoint union of two chains (Γ_1 or Γ_2 could be empty), let Δ_1, Δ_2 be the corresponding (absolute values of) the determinants ($\Delta(\emptyset) = 1$ by definition). In our situation we have only 2 bonus vertices, namely the ends of the chain Γ. Therefore

$$a_j = \frac{1}{\Delta}(\Delta_1 + \Delta_2) = \frac{\Delta_1\Delta_2}{\Delta}(\frac{1}{\Delta_1} + \frac{1}{\Delta_2}).$$

This is Case 1 of Fig.3.

(3.2.2). Let Γ be a graph having a single fork at a vertex v_j and suppose that $\Gamma - v_j = \Gamma_1 + \Gamma_2 + \Gamma_3$, and $\Delta_i = \Delta(\Gamma_i)$ for $i = 1, 2, 3$. In order for Γ to correspond to a log terminal (resp. log canonical) singularity one should have $a_j > 0$ (resp. $a_j \ge 0$). In this situation we have 3 bonus vertices, namely the simple ends of Γ_1, Γ_2, Γ_3 and 1 penalty vertex which is v_j itself. Therefore, by (3.1.10) one has

$$a_j = \frac{1}{\Delta}(\Delta_1\Delta_2 + \Delta_2\Delta_3 + \Delta_3\Delta_1 - \Delta_1\Delta_2\Delta_3) =$$
$$= \frac{\Delta_1\Delta_2\Delta_3}{\Delta}(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \frac{1}{\Delta_3} - 1).$$

So this is a log terminal singularity in the cases

(3.2.2.1). $(\Delta_1, \Delta_2, \Delta_3) = (2, 2, n), \quad n \ge 2$

(3.2.2.2). $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 3)$

(3.2.2.3). $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 4)$

(3.2.2.4). $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 5)$

and a log canonical (but not log terminal) singularity in the cases

(3.2.2.5). $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 6)$

(3.2.2.6). $(\Delta_1, \Delta_2, \Delta_3) = (2, 4, 4)$

(3.2.2.7). $(\Delta_1, \Delta_2, \Delta_3) = (3, 3, 3)$

This gives Cases 2 and 6 of Fig.3.

(3.2.3). Now let Γ be a graph with a single fork at the vertex v_j and suppose that $\Gamma - v_j = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, $\Delta_i = \Delta(\Gamma_i)$ for $i = 1, \dots, 4$.

Then

$$a_j = \frac{\Delta_1\Delta_2\Delta_3\Delta_4}{\Delta}(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \frac{1}{\Delta_3} + \frac{1}{\Delta_4} - 2)$$

and gives a log canonical singularity only if

(3.2.3.1) $(\Delta_1, \Delta_2, \Delta_3, \Delta_4) = (2, 2, 2, 2)$

This is Case 8 of Fig.3.

(3.2.4). In the case of graph Γ with a single fork at a vertex v_j, breaking up Γ into $N \ge 5$ subgraphs we get a non-log canonical singularity, because

$$a_j = \frac{\prod \Delta_i}{\Delta}(\sum_{i=1}^{N} \frac{1}{\Delta_i} - (N-2)) < 0$$

for $\Delta_i \ge 2$ and $N \ge 5$.

(3.2.5). Now suppose that we are in the situation of Fig.1 of a graph Γ with at least 2 forks, one of them at the vertex v_j. Suppose that $\Gamma - v_j = \Gamma_1 + \Gamma_2 + \Gamma_3$, and let Δ_1, Δ_2, Δ_3, Δ_A, Δ_B be the corresponding determinants. Then by (3.1.10),

$$a_j = \frac{\Delta_1\Delta_2\Delta_3}{\Delta}(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \frac{1 - (\Delta_A - 1)(\Delta_B - 1)}{\Delta_3} - 1).$$

This is nonnegative (actually, equal to zero) only in the case

$$\Delta_1 = \Delta_2 = \Delta_A = \Delta_B = 2.$$

By (3.1.10), this is also the sufficient condition for Γ to give a log canonical singularity. This is Case 7.

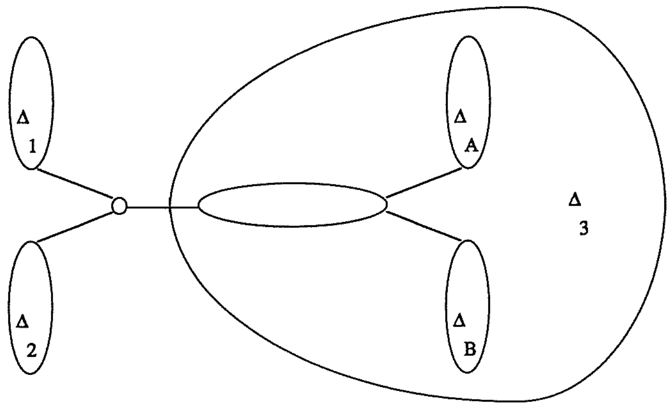


Fig.1

(3.2.6). Using (3.1.10) one can easily show that in the graphs of Fig.2 the marked vertices have negative log discrepancies, hence these graphs define non-log canonical singularities.

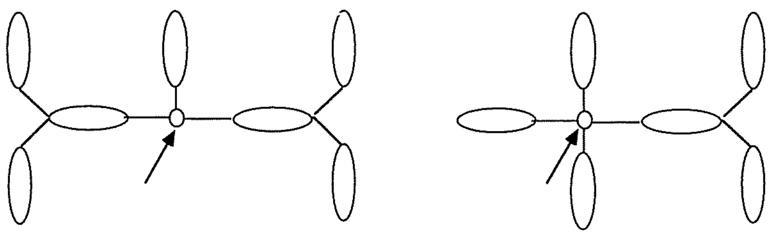


Fig.2

3.2.7 Lemma. *If Γ corresponds to a log terminal (log canonical) singularity then Γ is one of the graphs listed in (3.2.1–2.5).*

1st proof. (3.2.5) gives the general rule for what happens to a log discrepancy when we add an additional fork: the term, denote it by T, that corresponds to the part of the graph after the new fork is changed to a number

$$T \cdot (\Delta_A - \Delta_B(\Delta_A - 1))$$

with the corresponding $\Delta_A, \Delta_B \geq 1$. The other terms don't change.

Therefore, starting from (3.2.3), (3.2.4) or (3.2.6), adding a fork always gives a negative log discrepancy.

2nd proof. By (3.1.3) the subgraph $\Gamma' \subset \Gamma$ also defines a log canonical singularity. Therefore Γ cannot have subgraphs as in (3.2.4) or (3.2.6). □

(3.2.8). Note that Case 8 is essentially a subcase of 7.

3.3. The case $B \neq \emptyset$.

(3.3.1). In addition to the restrictions of (3.2) we have to consider additional penalties for the connections with $f_*^{-1}B$. Now it is an easy excercise to get the remaining Cases of Fig.3.

(3.3.2). From Fig.3 one can see that the minimal resolution is a good resolution for $K+B$. Note that in Case 9 with a chain containing a single vertex v_1, the curves corresponding to the black vertices do not intersect E_1. Otherwise, a single blow up gives a log discrepancy $a_2 = -1$.

(3.3.3). Note that in the Case 9 of Fig.3 all the discrepancies are zero because we have neither bonuses nor penalties.

(3.3.4). The index of a rational singularity, i.e. the least natural number N such that NK_X is a Cartier divisor, is at the same time the least common denominator of all the log discrepancies a_j. One can easily see that in the Cases 6–8 indices are 2,3,4 or 6.

3.4. Final remarks.

(3.4.1). Note that the only restriction on the unmarked weights on Fig.3 is that the quadratic form of the whole graph Γ should be negative definite. This is essential only in Cases 6–8 (where at least one weight should be > 2), and also in Case 5 (where either all weights are at least two and at least one at least three; or there are two vertices, one of them has weight one and the other has weight at least five).

An easy case by case check shows that in Cases 1–3 and 6-10 any (contractible) graph defines a rational singularity, so by [Artin66] a configuration can be contracted to a normal singular point. In cases 4–5 if the quadratic form is negative definite, then a configuration can be contracted in the analytic situation. In the algebraic situation this is a necessary condition (but not sufficient).

(3.4.2). Our method allows one in principle to classify log terminal or log canonical surface singularities $(X, K+B)$ when B may have fractional coefficients with denominators $2, 3, \dots$, if this should turn out to be necessary. There will be a large number of new cases.

K+B is log terminal, B is reduced

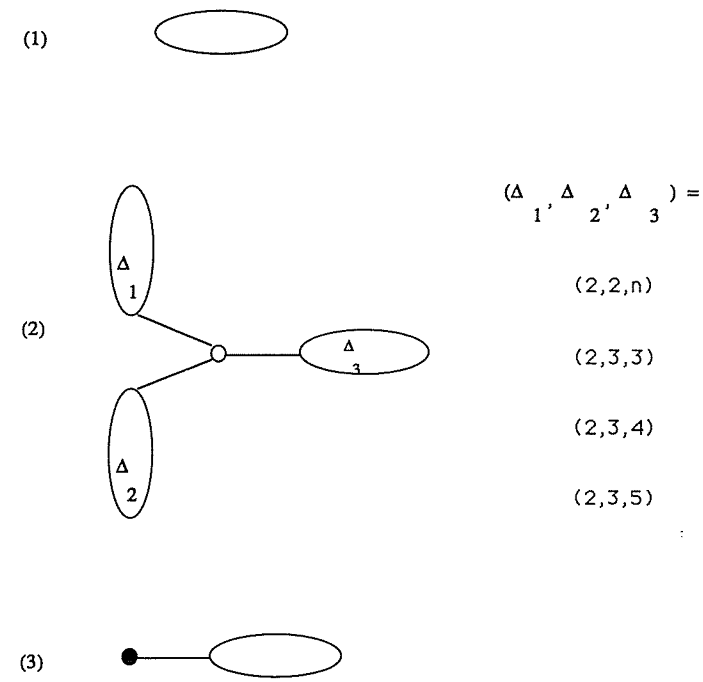
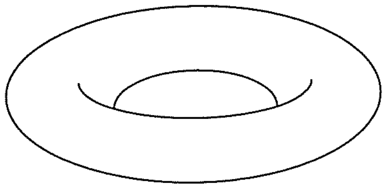


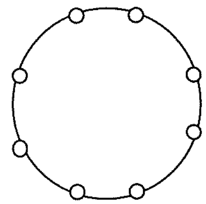
Fig.3, beginning

K+B is log canonical but not log terminal, B is reduced

(4)



(5)



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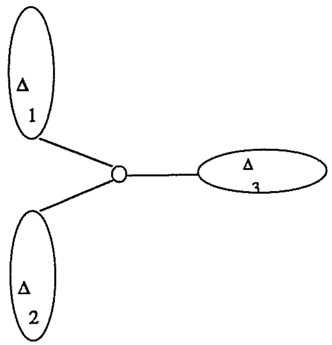


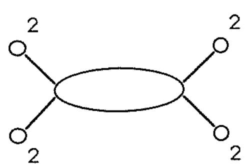
$(\Delta_1, \Delta_2, \Delta_3) =$

(3,3,3)

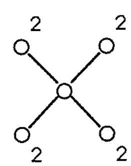
(2,4,4)

(2,3,6)

(7)

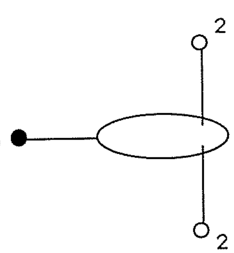


(8)



(9)

(10)



4. TERMINATION OF CANONICAL FLIPS

JÁNOS KOLLÁR and KENJI MATSUKI

The aim of this chapter is to study flops and flips for terminal and canonical threefolds. First we prove the basic finite generation theorem of [Reid83]. The second main result is termination of flips (and flops) for canonical pairs (X, D) (4.10). We start with some general results that hold for arbitrary schemes.

4.1 Definition. Let X be a normal scheme. A *small modification* of X is a proper birational morphism $f : Y \to X$ such that Y is normal and the exceptional set of f has codimension ≥ 2. We usually exclude the trivial case $Y \cong X$.

The following proposition relates projective small modifications to the divisor class group Weil(X) (cf. (16.3.1)).

4.2 Proposition. *[Kawamata88,3.1] Let X be a normal scheme and let D be a Weil divisor on X (not necessarily effective). The following two statements are equivalent:*

(4.2.1) $\sum_{m=0}^{\infty} \mathcal{O}_X(mD)$ is a finitely generated $\mathcal{O}_X$-algebra.

(4.2.2) There is a small modification $f : Y \to X$ such that D', the birational transform of D on Y, is $\mathbb{Q}$-Cartier and f-ample.

Furthermore f is nontrivial iff no positive multiple of D is Cartier.

Proof. Assume that $f : Y \to X$ exists. Let $C \subset Y$ be the exceptional set. First we claim that

$$f_*\mathcal{O}_Y(mD') = \mathcal{O}_X(mD) \quad \text{for } m \geq 0. \tag{4.2.3}$$

It is always true that $f_*\mathcal{O}_Y(mD') \subset \mathcal{O}_X(mD)$. Let $C \subset Y$ be the exceptional set of f. Let $s : \mathcal{O}_X \to \mathcal{O}_X(mD)$ be a section. We can pull it back to a section

$$s : \mathcal{O}_{Y-C} \to \mathcal{O}_{Y-C}(mD').$$

S. M. F.
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