Изв. Акад. Наук СССР
Сер. Матем. Том 52(1988), № 6

Math. USSR Izvestiya
Vol. 33(1989), No. 3

FRACTIONAL INDICES OF LOG DEL PEZZO SURFACES

UDC 512.774

V. A. ALEKSEEV

ABSTRACT. The *fractional index* of a (possibly singular) $\mathbb{Q}$-Gorenstein del Pezzo surface X is the greatest rational number r such that $-K_X \equiv rH$, where H is a primitive Cartier divisor. This paper describes the set of values taken by fractional indices of del Pezzo surfaces with log terminal singularities.

Bibliography: 8 titles.

Introduction

Let X be a log del Pezzo surface, that is, a singular normal complex surface with ample anticanonical class and log terminal singularities (see §1 for precise definitions). These surfaces are a generalization of a very well studied class of surfaces, the usual del Pezzo surfaces; their classification is interesting in its own right, and may also be useful in the theory of minimal models of algebraic 3-folds. Among algebraic 3-folds, the class that can naturally be thought of as parallel to these is the class of Fano 3-folds with terminal singularities; one can expect the difficulties arising in the study of these two classes of varieties to have something in common.

The *index* of a nonsingular del Pezzo surface S is the greatest natural number r such that the anticanonical divisor $-K_S$ is divisible by r in $\operatorname{Pic} S$. It is well known that $r = 3$ if $S \cong \mathbb{P}^2$, $r = 2$ if $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $r = 1$ otherwise. On a log del Pezzo surface X, the anticanonical Weil divisor $-K_X$ may not be a Cartier divisor, but in the group $\operatorname{Pic} X \otimes \mathbb{Q}$ of $\mathbb{Q}$-Cartier divisors it is natural to write $-K_X = rH$, where H is an ample Cartier divisor primitive in $\operatorname{Pic} X$ and r a rational number. In this case, the number $r = r(X)$ is called the *fractional index* of X. There arises the interesting question of what the set

$$R = \{r(X) \mid X \text{ is a log del Pezzo surface}\}$$

looks like. In this paper, we give the following description of R.

THEOREM 4.3. *The set R has the following accumulation points*: 0 *and* $1/m$ *for any natural number m. All of these points are limit points from above and not from below.*

Moreover, for any natural number m, we can choose a sufficiently small punctured neighborhood $\mathring{Q}_m = \{x \in \mathbb{R} \mid 0 < |x - 1/m| < \varepsilon_m\}$ in such a way that all log del Pezzo surfaces X with $r(X) \in \mathring{Q}_m$ can be classified explicitly.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14J26; Secondary 14J05, 14J17, 14J25, 14E30, 51F15.

As intermediate results, we prove theorems on the boundedness of the rank of the Picard lattice of the minimal resolution of singularities of X, Theorems 2.3, 2.4 and 2.5. These boundedness results stem from Theorem 2.2, which is due to V. V. Nikulin, and is based on methods developed in the theory of discrete reflection groups in Lobachevsky space in papers of Nikulin, Vinberg, Prokhorov and Khovanskiĭ.

The layout of the paper is as follows. In §1 we give precise definitions and basic information on log del Pezzo surfaces. In §2 we state the boundedness theorems. In §3 we give the bounds which we need in the proofs of these theorems; these make use of results of V. V. Nikulin on log terminal Lannér graphs. §4 is concerned with fractional indices proper.

I would like to express my gratitude to V. V. Shokurov for setting the problem, to V. V. Nikulin for allowing me to make use of as yet unpublished material, and to V. A. Iskovskikh for interest in this work.

§1. Basic facts on log del Pezzo surfaces

Let X be a normal algebraic surface; the canonical Weil divisor K_X of X is defined.

DEFINITION 1.1. We say that X has at worst *log terminal singularities* if the following conditions are satisfied:

(i) Some multiple NK_X of the canonical divisor of X is a Cartier divisor.

(ii) Let $\pi: Y \to X$ be a minimal resolution of singularities and $NK_Y = \pi^*(NK_X) + \sum \alpha_i F_i$ the natural formula, where F_i are the exceptional prime divisors. Rewrite this formula in $\operatorname{Pic} Y \otimes \mathbb{Q}$ in the form

$$K_Y = \pi^* K_X + \sum \alpha_i F_i;$$

then all the α_i should be greater than -1.

It is not hard to prove that all 2-dimensional log terminal singularities are rational, and that condition (ii) holds for any other resolution. All the log terminal singularities are listed in [4] and [5], in [5] in a purely arithmetic way, starting from condition (ii). The exceptional curves introduced by the blow-ups form one of the graphs A_n, D_n, E_6, E_7 or E_8.

DEFINITION 1.2. A normal algebraic surface X is a *log del Pezzo surface* if

(i) X has only log terminal singularities, and

(ii) some multiple $-NK_X$ of the anticanonical divisor is an ample Cartier divisor.

In [1], I proved the following simple proposition.

PROPOSITION 1.3. (i) *Any log del Pezzo surface X is rational.*

(ii) *Let $\pi: Y \to X$ be a minimal resolution of singularities. Then the exceptional curves of Y (that is, the irreducible curves with negative selfintersection) are the exceptional curves of the resolution morphism π, together with (-1)-curves. There are only finitely many of them.*

Since the singularities of X are normal, $\pi^*: \operatorname{Pic} X \to \operatorname{Pic} Y$ is an inclusion of groups. Hence $\operatorname{Pic} X$ is a torsion-free Abelian group of finite rank, and the difference $\rho(Y) - \rho(X)$ is equal to the number of exceptional curves of π.

We have a vector space $\operatorname{Pic}_{\mathbb{Q}} X = \operatorname{Pic} X \otimes \mathbb{Q}$ and an integral lattice $\operatorname{Pic} X$ in it. Since all 2-dimensional rational singularities are $\mathbb{Q}$-factorial (that is, any Weil divisor has a multiple which is a Cartier divisor), any Weil divisor D can, up to linear equivalence, be viewed as an element of $\operatorname{Pic}_{\mathbb{Q}} X$. The lattice $\operatorname{Pic} Y$ and the vector space $\operatorname{Pic}_{\mathbb{Q}} Y$ are defined similarly. All of these groups are naturally equipped with intersection forms, which are related by morphisms

$$\pi^*: \operatorname{Pic}_{\mathbb{Q}} X \to \operatorname{Pic}_{\mathbb{Q}} Y \quad \text{and} \quad \pi_*: \operatorname{Pic}_{\mathbb{Q}} Y \to \operatorname{Pic}_{\mathbb{Q}} X$$

DEFINITION 1.4. The *index*, or *Gorenstein index*, of a log del Pezzo surface is the smallest natural number N such that $-NK_X \in \operatorname{Pic} X$.

DEFINITION 1.5. The *fractional index* of a log del Pezzo surface is the maximal positive rational number r such that $-\frac{1}{r}K_X \in \operatorname{Pic} X$, that is, $-K_X = rH$, where H is a primitive element of $\operatorname{Pic} X$.

Obviously $r = M/N$, where M is a natural number.

We use the following notation for the exceptional curves of Y: write F_i for the exceptional curves of π and E_j for the (-1)-curves.

DEFINITION 1.6. The *DP-coefficients* of a log del Pezzo surface X are the following positive rational numbers:

$$\eta_j = \eta(E_j) = -\pi^* K_X \cdot E_j = 1 + \sum_i \alpha_i F_i \cdot E_j,$$

where the E_j are the (-1)-curves.

PROPOSITION 1.7. *If* $\rho(Y) > 2$ *then* $r = \gcd(\eta_j)$.

PROOF. First of all, $-\frac{1}{r}K_X$ is a Cartier divisor if and only if $-\frac{1}{r}\pi^*K_X$ is. Secondly, the lattice $\operatorname{Pic} Y$ is unimodular, so that for this it is necessary and sufficient that $-\frac{1}{r}\pi^*K_X \cdot v \in \mathbb{Z}$ for every $v \in \operatorname{Pic} Y$. Finally, if $\rho(Y) > 2$ then $\operatorname{Pic} Y$ is generated by the classes of exceptional curves, and $-\pi^*K_X \cdot F_i = 0$ and $-\pi^*K_X \cdot E_j = \eta_j$.

§2. Boundedness theorems

2.1. PROPERTY $DP(\varepsilon)$. Let ε be some positive real number. We say that a log del Pezzo surface *satisfies condition* $DP(\varepsilon)$ if all of its DP-coefficients satisfy $\eta_j \geq \varepsilon$.(1)

From this condition, we deduce various kinds of boundedness theorems on the rank $\rho(Y)$ of the Picard group of the minimal resolution of singularities.

Since all the subsequent arguments will be in terms of graphs, we introduce some notation we will need. By a graph we mean a nonoriented graph with finitely many vertices; we denote graphs by capital letters such as Γ. We have in particular the standard graphs A_n, D_n, E_6, E_7, E_8; and $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$.

By a *weighted* graph Γ, we mean a graph where each vertex is given a weight p_i. The graphs A_n, D_n, E_6, E_7, E_8, $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ have a standard weighting in which every vertex has weight $p_i = -2$. With a weighted graph $\Gamma = \{v_1, \dots, v_k\}$ we associate in a natural way a quadratic form by setting $v_i^2 = p_i$ and $v_i \cdot v_j$ equal to the number of edges of Γ joining v_i and v_j.

We say that a weighted graph Γ is *elliptic, parabolic* or *hyperbolic* if the associated quadratic form has signature respectively $(0, k)$, $(0, k-1)$ or $(1, k-1)$. A weighted graph Γ is *Lannér* if it is hyperbolic and no proper subgraph of Γ is hyperbolic.

The proofs of our boundedness theorems are based on the following theorem due to V. V. Nikulin [6]. Let X be a log del Pezzo surface and $\pi\colon Y \to X$ the minimal resolution of singularities. With Y we associate the weighted graph of exceptional curves as follows: to each exceptional curve C_i we assign a vertex of weight $p_i = C_i^2$, and we join two vertices C_i and C_j by $C_i \cdot C_j$ edges. The quadratic form corresponding to this graph is simply the intersection form $(C_i \cdot C_j)$ of curves on Y.

THEOREM 2.2. *Suppose that the following two conditions hold*:

(i) *Every Lannér subgraph* Γ' *of* Γ *has at most* l *vertices.*

(1)*Translator's note.* An obvious reformulation is $K_Y + \lambda(-\pi^*K_X)$ is nef, where $\lambda = 1/\varepsilon$; since $-K_X$

(ii) *For any connected elliptic subgraph* Γ' *of* Γ *with* n *vertices, the number of pairs of vertices of* Γ' *at distance* $d \leq l - 3$ *is bounded by* $c_1 n$, *and the number of pairs at distance* $l - 3 < d \leq 2l - 3$ *is bounded by* $c_2 n$.

Then

$$\rho(Y) \leq 96(c_1 + c_2/3) + 69.$$

Instead of condition (ii), we could assume the following stronger condition (ii)′ which implies (ii).

(ii)′ *For any connected elliptic subgraph* Γ' *of* Γ, *the number of vertices of* Γ' *at distance* $d \leq l - 3$ *from any fixed vertex is at most* c_1, *and the number at distance* $l - 3 < d \leq 2l - 3$ *is at most* c_2.

In the following section, we will prove that if a log del Pezzo surface X satisfies condition $DP(\varepsilon)$ then $l \leq 7 + 16/\varepsilon$, $c_1 \leq 16 + 44/\varepsilon$, and $c_2 \leq 17 + 44/\varepsilon$ (see Theorems 3.13 and 3.15).

The next theorem follows at once from these bounds and Theorem 2.2.

THEOREM 2.3. *Suppose that a log del Pezzo surface satisfies condition* $DP(\varepsilon)$. *Then the rank* $\rho(Y)$ *of the Picard group of the minimal resolution of singularities satisfies* $\rho(Y) \leq F(1/\varepsilon)$, *where* F *is a linear function.*

It is easy to write out this function explicitly from the bounds we have given; it is clear that this is far from being the best bound. The essential point is just that the bound is linear in $1/\varepsilon$. We will show in Example 4.5 below that this bound cannot be made better than linear.

As corollaries of Theorem 2.3 we deduce the following boundedness theorems.

THEOREM 2.4. $\rho(Y) \leq F(1/r)$, *where* F *is a linear function and* r *the fractional index of the log del Pezzo surface* X.

THEOREM 2.5. $\rho(Y) \leq F(k)$, *where* F *is a linear function and* k *the index of the log del Pezzo surface* X.

The proof follows automatically from the following relations:

$$\min(\eta_j) \geq r = \gcd(\eta_j) = M/k \geq 1/k.$$

REMARK 2.6. The boundedness theorem (2.5) was proved by V. V. Nikulin in [7] with the function $F(k) \sim k^{7/2}$.

We give another much simpler proof of Theorem 2.3 for $\varepsilon = 1/2$.

PROPOSITION 2.7. *Suppose that a log del Pezzo surface satisfies condition* $DP(1/2)$. *Then* $\rho(Y) \leq 10$.

PROOF. We have $K_Y = \pi^* K_X + \sum \alpha_i F_i$. Introduce the "reflected" canonical class $\overline{K} = \pi^* K_X - \sum \alpha_i F_i$. We will prove that

$$-\overline{K} = K_Y + 2\pi^*(-K_X)$$

is nef. Therefore $(-\overline{K})^2 \geq 0$; on the other hand, $K_Y^2 = (-\overline{K})^2$, and by Noether's formula $\rho(Y) = 10 - K_Y^2 \leq 10$.

As I proved in [1], the Mori-Kleiman cone $\mathrm{NE}(Y)$ of effective cycles is generated by the classes of the exceptional curves F_i and E_j. Hence for $-\overline{K}$ to be nef, it is necessary and sufficient that $-\overline{K} \cdot F_i \geq 0$ and $-\overline{K} \cdot E_j \geq 0$. We have $-\overline{K} \cdot F_i = K_Y \cdot F_i = -F_i^2 - 2 \geq 0$ in view of the minimality of the resolution of singularities. Moreover,

$$1 = -K_Y \cdot E_j, \qquad \eta_j = -\pi^* K_X \cdot E_j, \qquad -\overline{K} \cdot E_j$$

form an arithmetic progression, and since $\eta_j \geq 1/2$ we get $-\overline{K} \cdot E_j \geq 0$.

§3. Bounds on l, c_1 and c_2

In this section we prove the bounds on l, c_1 and c_2 of Theorem 2.2, assuming that X satisfies $DP(\varepsilon)$.

Let us add to our list of notation, by agreeing on how we will draw a weighted graph. The weights of a vertex will always be negative, and so we omit the minus sign. Vertices with $p_i = -1$ will be referred to (and drawn) as *white* vertices; those with $p_i \leq -2$ as *black* vertices. By a *black graph*, we mean one consisting entirely of black vertices.

We now model in the language of weighted graphs the definitions of the coefficients α_i appearing in the formula $K_Y = \pi^*K_X + \sum \alpha_i f_i$, of the DP-coefficients, and of blow-ups of a surface.

DEFINITION 3.1. Let $\Gamma' = \{f_1, \dots, f_k\}$ be an elliptic graph. The *coefficients* of Γ' are the rational numbers $\alpha_i = \alpha(\Gamma', f_i)$ which are uniquely determined by the system of linear equations

$$\sum_{i=1}^{k} \alpha_i f_i \cdot f_j = -p_j - 2 \qquad \text{for } j = 1, \dots, k.$$

(If Γ' is the graph corresponding to a system of nonsingular rational curves on a surface Y, and $\pi\colon Y \to X$ is the contraction of these curves to points, then $K_Y = \pi^*K_X + \sum \alpha_i f_i$. This follows from the adjunction formula for the genus of the curves f_j.)

DEFINITION 3.2. An elliptic graph Γ is *log terminal* if it has every coefficient $\alpha_i > -1$.

DEFINITION 3.3. Let Γ be a weighted graph and $\Gamma' \subset \Gamma$ an elliptic subgraph which contains all the black vertices, and possibly some white ones: $\Gamma' = \{f_1, \dots, f_s\}$, $\Gamma = \Gamma' \cup \{e_1, \dots, e_t\}$. We define DP-coefficients for the remaining white vertices e_j as follows:

$$\eta_j = \eta(\Gamma, \Gamma'; e_j) = 1 + \sum_i \alpha_i(\Gamma') f_i \cdot e_j.$$

(If Γ' and π are as above, and e_j is a (-1)-curve, then $\eta_j = -\pi^*K_X \cdot e_j$.)

DEFINITION 3.4. Consider a graph Γ and a subgraph Γ'; we say that the pair (Γ, Γ') *satisfies condition $DP(\varepsilon)$* if Γ and Γ' are as in Definition 3.3, and all the DP-coefficients $\eta_j \geq \varepsilon$.

We will say, for short, that a graph Γ *satisfies $DP(\varepsilon)$*, to mean that the above condition holds, where Γ' is assumed to be the graph consisting of all the black vertices.

DEFINITION 3.5. By the *blow-up* of a weighted graph Γ in a complete subgraph $\Gamma_1 = \{v_1, \dots, v_k\}$, we mean the following transformation: the weight of each vertex of Γ_1 decreases by 1, that is, $p_i' = p_i - 1$, and the number of edges joining v_i and v_j decreases by 1, that is, $v_i' \cdot v_j' = v_i \cdot v_j - 1$. A new white vertex e is added to the graph, attached by simple edges to the vertices $\{v_1', \dots, v_k'\}$. As a general rule, we only blow up a vertex $\Gamma_1 = \{v_1\}$ or an edge $\Gamma_1 = \{v_1, v_2\}$. (This is how the graph of smooth curves on a surface changes under a blow-up.)

A sequence of inverse transformations of blow-ups is called a *morphism*.

DEFINITION 3.6. A weighted graph Γ' is called a *predecessor* of Γ if there exists a morphism $\sigma\colon \Gamma \to \Gamma'$. In particular, the graph Γ is its own predecessor.

Note that if a weighted graph Γ is Lannér then any predecessor of Γ is again Lannér.

Preliminary lemmas.

LEMMA 3.7. (i) *Suppose that* $\Gamma = \Gamma' \cup \{v_{k+1}\}$ *is an elliptic graph with* $\Gamma' = \{v_1, \dots, v_k\}$ *and such that* v_{k+1} *is a white vertex having DP-coefficient* $\eta(\Gamma, \Gamma'; v_{k+1}) > 0$. *Then the coefficients of the graphs* Γ *and* Γ' *are related by the following inequality:*

$$\alpha(\Gamma, v_i) > \alpha(\Gamma', v_i) \qquad \textit{for } i = 1, \dots, k.$$

(ii) *Let* $\Gamma = \Gamma' \cup \{v_{k+1}\}$ *be a black elliptic graph. Then*

$$\alpha(\Gamma, v_i) \leq \alpha(\Gamma', v_i) \qquad \textit{for } i = 1, \dots, k.$$

(iii) *Let* $\Gamma = \{v_1, \dots, v_k\}$ *be an elliptic graph and* Γ' *a weighted graph that differs from* Γ *only in the weight at the vertex* v_1, *with* $p_1' < p_1$. *Then*

$$\left.\begin{aligned} \alpha(\Gamma, v_1) > -1 &\implies \alpha(\Gamma', v_i') < \alpha(\Gamma, v_i) \\ \alpha(\Gamma, v_1) = -1 &\implies \alpha(\Gamma', v_i') = \alpha(\Gamma, v_i) \\ \alpha(\Gamma, v_1) < -1 &\implies \alpha(\Gamma', v_i') > \alpha(\Gamma, v_i) \end{aligned}\right\} \quad \textit{for } i = 1, \dots, k.$$

(iv) *Let* Γ *be a black elliptic graph and* Γ' *a weighted graph that differs from* Γ *only in the presence of an extra edge,* $v_1' \cdot v_2' = v_1 \cdot v_2 + 1$. *Then* $\alpha(\Gamma', v_i') \leq \alpha(\Gamma, v_i)$ *for* $i = 1, \dots, k$.

(The proof of this is actually already contained in [7].)

PROOF. (i) Set $\alpha(\Gamma', v_{k+1}) = 0$ and compare the two systems of linear equations

$$\sum_{i=1}^{k+1} \alpha(\Gamma', v_i) v_i \cdot v_j = -p_j - 2 \qquad \text{for } j = 1, \dots, k,$$

$$\sum_{i=1}^{k+1} \alpha(\Gamma', v_i) v_i \cdot v_{k+1} = -p_{k+1} - 2 + \eta(\Gamma, \Gamma'; v_{k+1})$$

and

$$\sum_{i=1}^{k+1} \alpha(\Gamma, v_i) v_i \cdot v_j = -p_j - 2 \qquad \text{for } j = 1, \dots, k+1.$$

Subtracting one from the other gives

$$\sum_{i=1}^{k+1} (\alpha(\Gamma', v_i) - \alpha(\Gamma, v_i)) v_i \cdot v_j = 0 \qquad \text{for } j = 1, \dots, k,$$

$$\sum_{i=1}^{k+1} (\alpha(\Gamma', v_i) - \alpha(\Gamma, v_i)) v_i \cdot v_{k+1} = \eta(\Gamma, \Gamma'; v_{k+1}) > 0.$$

Let $V = (v_i \cdot v_j)$ be the matrix of the intersection form. All the entries of the inverse matrix V^{-1} are negative [2]. Hence $\alpha(\Gamma', v_i) - \alpha(\Gamma, v_i) < 0$ for $i = 1, \dots, k+1$.

(ii) In the same way as in (i), we have

$$\sum_{i-1}^{k+1} (\alpha(\Gamma', v_i) - \alpha(\Gamma, v_i)) v_i \cdot v_j = 0 \qquad \text{for } j = 1, \dots, k,$$

$$\sum_{i=1}^{k+1} (\alpha(\Gamma', v_i) - \alpha(\Gamma, v_i)) v_i \cdot v_{k+1} = p_{k+1} + 2 + \sum_{i=1}^{k} \alpha(\Gamma', v_i) \cdot v_{k+1} \leq 0,$$

since $p_{k+1} \le -2$ and $\alpha(\Gamma', v_i) \le 0$. The final statement follows from the fact that the graph Γ' is black, and its coefficients satisfy the system of equations

$$\sum \alpha(\Gamma', v_i) v_i \cdot v_j = -p_j - 2 \ge 0.$$

This proves the required inequality.

(iii) In an entirely similar way we have

$$\sum_{i=1}^{k} (\alpha(\Gamma', v_i') - \alpha(\Gamma, v_i)) v_i' \cdot v_j' = 0 \qquad \text{for } j = 2, \dots, k,$$

$$\sum_{i=1}^{k} (\alpha(\Gamma', v_i') - \alpha(\Gamma, v_i)) v_i' \cdot v_1' = (1 + \alpha_1)(p_1 - p_1'),$$

which gives all the inequalities.

(iv) We have

$$\sum_{i=1}^{k} (\alpha(\Gamma', v_i') - \alpha(\Gamma, v_i)) v_i' \cdot v_j' = 0 \qquad \text{for } j \ne 1, 2,$$

$$\sum_{i=1}^{k} (\alpha(\Gamma', v_i') - \alpha(\Gamma, v_i)) v_i' \cdot v_1' = -\alpha_2 \ge 0,$$

$$\sum_{i=1}^{k} (\alpha(\Gamma', v_i') - \alpha(\Gamma, v_i)) v_i' \cdot v_2' = -\alpha_1 \ge 0.$$

LEMMA 3.8. (i) *Let* $\Gamma = \{e_1, \dots, e_s, f_1, \dots, f_t\}$ *be a parabolic graph and* $\Gamma' = \{f_1, \dots, f_t\} \subset \Gamma$ *a subgraph such that* Γ, Γ' *satisfies Definition* 3.3, *with* η_j *the corresponding DP-coefficients. Let* $C = \sum a_j e_j + \sum b_i f_i$ *be the minimal effective cycle with integer coefficients such that* $C \cdot e_j = C \cdot f_i = 0$. *It is easy to show that* C *exists; we call* C *the* ***fundamental cycle*** *of* Γ.

Then

$$\sum a_j \eta_j = 2(1 - p_a C), \quad \textit{where } p_a C \textit{ is the arithmetic genus.}$$

(ii) *Let* Γ *be an elliptic graph and* Γ' *a log terminal subgraph; let* C *be the fundamental cycle for* Γ, *that is, the minimal effective cycle such that* $C \cdot e_j \le 0$ *and* $C \cdot f_i \le 0$. *The existence of such a cycle is proved in* [3]. *The remaining conditions are as above. Then*

$$\sum a_j \eta_j < 2(1 - p_a C).$$

(*Note that* $p_a C \ge 0$ *in* (i) *and* (ii).)

(iii) *Let* Γ_0 *and* C_0 *be an arbitrary graph and cycle, and* $\sigma \colon \Gamma \to \Gamma_0$ *a morphism; write* $C = \sigma^* C_0$, *and suppose the remaining conditions are as above. Then*

$$\sum a_j \eta_j \le 2(1 - p_a C_0) + \operatorname{const}(C_0).$$

PROOF. Although the proof we give below is purely arithmetic in nature, and is applicable to arbitrary graphs for which one can define the canonical class as a linear functional, we will work in the only situation of interest to us, when Γ is the graph of exceptional curves on a surface; this simplifies the notation and allows us to omit unnecessary definitions, for example of the arithmetic genus.

(i) Let $f\colon Y \to X$ be the contraction of the curves of Γ'. Then $K_Y = f^*K_X + \sum \alpha_i f_i$; and

$$\sum a_j\eta_j = -\pi^*K_X \cdot C = \left(-K_Y + \sum \alpha_i f_i\right) \cdot C$$
$$= 2(1 - p_a C) + \left(C + \sum \alpha_i f_i\right) \cdot C = 2(1 - p_a C).$$

(ii) In the preceding formula, note that $(C + \sum \alpha_i f_i) \cdot C \le 0$; this follows from the definition of the fundamental cycle and the fact that $C \ge \sum f_i$ and $\sum \alpha_i f_i < -\sum f_i$ by the log terminal property of Γ'.

(iii) We have

$$\left(C + \sum \alpha_i f_i\right) \cdot C = \left(C + \sum \alpha_i f_i\right) \cdot \sigma^* C_0$$
$$= \left(C_0 + \sigma_* \sum \alpha_i f_i\right) \cdot C_0 \le C_0^2 - \sum g_i C_0,$$

where the g_i are the vertices of Γ_0 such that $g_i C_0 < 0$.

COROLLARY 3.9. *Suppose that the weighted graph Γ satisfies condition $DP(\varepsilon)$; then every subgraph $\Gamma_1 \subset \Gamma$ also satisfies $DP(\varepsilon)$.*

The proof follows from 3.7(ii) and the definition of DP-coefficients.

COROLLARY 3.10. *Suppose that the weighted graph Γ satisfies $DP(\varepsilon)$ for some $\varepsilon > 0$, and that the subgraph of black vertices is log terminal. Let Γ' be a weighted graph that is a predecessor of Γ, and $\Gamma'' \subset \Gamma'$ the subgraph of black vertices. Then Γ'' cannot contain $\widetilde{\mathrm{A}}_n$, $\widetilde{\mathrm{D}}_n$ or $\widetilde{\mathrm{E}}_n$ as subgraphs, nor have multiple edges.*

PROOF. Suppose that Γ' contains a black subgraph Γ_1' of one of the types $\widetilde{\mathrm{A}}_n$, $\widetilde{\mathrm{D}}_n$ or $\widetilde{\mathrm{E}}_n$, and let $\Gamma_1 \subset \Gamma$ be the inverse image of Γ_1' under the morphism $\sigma\colon \Gamma \to \Gamma'$. Let $C = \sum a_j e_j + \sum b_i f_i$ be the fundamental cycle of the graph Γ_1 as in Lemma 3.8.

If Γ_1' has weight -2 at each vertex then $p_a C = 1$ and $\sum a_j n_j = 0$ by Lemma 3.8(i). But since Γ_1 is parabolic, it must contain at least one white vertex and so $\sum a_j n_j > 0$, a contradiction.

If Γ_1' has weight less than -2 at some vertex then Γ_1 is elliptic. By the description of log terminal singularities, Γ_1' cannot coincide with Γ_1, that is, it must contain at least one white vertex. It follows from Lemma 3.7(iii) that the sum $\sum a_j n_j$ must be less than in the preceding case, that is, $\sum a_j n_j < 0$. On the other hand, we have $\eta_j \ge \varepsilon > 0$, and hence $\sum a_j n_j > 0$, a contradiction.

Next, an edge of multiplicity 2 is a subgraph $\widetilde{\mathrm{A}}_1$. An edge of multiplicity > 2 gives a contradiction a fortiori by Lemma 3.7(iv).

COROLLARY 3.11. *Suppose that a weighted graph Γ satisfies condition $DP(\varepsilon)$ for some $\varepsilon > 0$. Then for any weighted graph Γ' that is a predecessor of Γ, the black vertices form a disjoint union of graphs A_n, D_n and E_n.*

PROOF. Indeed, these are the only graphs that satisfy Corollary 3.10.

We now proceed to the bounds for the constants l, c_1 and c_2 of Theorem 2.2. For these, we make use of the following description of Lannér graphs given by V. V. Nikulin. The proof of the next theorem is contained in [7], (1.1.8).

THEOREM 3.12. *Let Γ be a Lannér graph satisfying condition $DP(\varepsilon)$ for some $\varepsilon > 0$. Then one of the following conditions holds:*

(i) *Γ has at most 3 vertices.*

(ii) *The weighted graph* Γ *contains a parabolic subgraph* Γ_{para} *whose fundamental cycle satisfies* $p_a(C_{\text{para}}) = 1$.

(iii) *The weighted graph* Γ *contains a parabolic subgraph* Γ_{para} *with* $p_a(C_{\text{para}}) = 0$, *and there exists a morphism* $\sigma\colon \Gamma \to G_2(1,1,b)$ *or* $\sigma\colon \Gamma \to G_3(1,1,b)$, *where* $G_2(1,1,b)$ *and* $G_3(1,1,b)$ *are the graphs corresponding to the quadratic forms*

$$\begin{pmatrix} -b & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -b & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \qquad \text{for } b \geq 1.$$

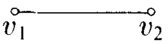
(iv) *There exists a morphism* $\sigma\colon \Gamma \to G(1,2;2)$ *or* $\sigma\colon \Gamma \to G(1,3;2)$, *where* $G(1,2;2)$ *and* $G(1,3;2)$ *are the graphs corresponding to the quadratic forms*

$$\begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}.$$

THEOREM 3.13. *Suppose that the weighted graph* Γ *satisfies* $DP(\varepsilon)$ *with* $\varepsilon > 0$. *Then the number of vertices of any Lannér subgraph of* Γ *in* $\widetilde{\Gamma}$ *does not exceed* $7 + 16/\varepsilon$.

PROOF. By Corollary 3.9, Γ_1 also satisfies $DP(\varepsilon)$. We consider one by one the various cases of Theorem 3.12. Case (i) does not concern us, since if $\varepsilon \leq 1$ then $3 < 7 + 16/\varepsilon$, and if $\varepsilon > 1$ then there are no graphs satisfying $DP(\varepsilon)$. Case (ii) does not occur by Lemma 3.8(i); one gets a contradiction exactly as in the proof of Corollary 3.10.

Case (iii). Thus we have a subgraph $\Gamma_{\text{para}} \subset \Gamma$, and for the morphism σ we have $\sigma\colon \Gamma_{\text{para}} \to O$, where O is the following graph:



Then the fundamental cycle C_{para} of Γ_{para} must be $C_{\text{para}} = \sigma^*(v_1 + v_2)$. Suppose that $C_{\text{para}} = \sum a_j e_j + \sum b_i f_i$, where the e_j are white vertices and the f_i black. The natural numbers a_j and b_i occuring in this expression will be called the *multiplicities* of the vertices. It is easy to see how these multiplicities change under blow-ups. Let $\pi\colon \Gamma' \to \Gamma''$ be a blow up of a weighted graph Γ'' in a complete subgraph Γ''' consisting of vertices $v_1, \dots, v_k$ with multiplicities $c_1, \dots, c_k$. Then the new vertex e will have multiplicity $c_1 + \cdots + c_k$, and the other multiplicities remain unchanged.

The idea of the present proof is to write the morphism $\sigma\colon \Gamma_{\text{para}} \to O$ in a convenient way as a sequence of blow-ups, and to keep track of the way in which the sum $A = \sum a_j$ of the multiplicities of the white vertices changes under the blow-ups.

LEMMA 3.14. *The morphism* $\sigma\colon \Gamma_{\text{para}} \to O$ *can be represented as the following composite of blow-ups*:

Step 1. *We blow up only vertices; moreover, the number of new black vertices appearing is at most* $4/\varepsilon$.

Step 2. *We first blow up an edge; then optionally the newly appeared vertex, then again optionally the most recently appeared vertex, and so on. The number of new vertices appearing in this step is at most* 6.

Step 3. *We blow up only black vertices which appeared in Step* 2.

Then Step 2 *and Step* 3 *may be repeated any number of times.*

PROOF OF THE LEMMA. We can obviously write the morphism as a composite of blow-ups such that first only vertices of multiplicity 1 appear, then of multiplicity 2, then of multiplicity 3, and so on. If the newly appeared vertices have multiplicity 1 then only vertices have been blown up; this is precisely what Step 1 consists of.

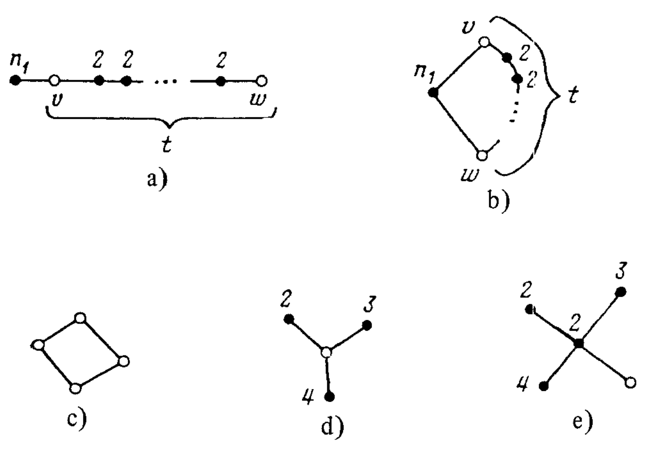


FIGURE 1, (a), (b), (c), (d), (e)

Consider the graph Γ_1 obtained on completion of this step. Γ_1 is a very simple type of weighted graph: it has simple edges forming a tree, with the weight at each vertex equal to the number of its neighbors. In particular, each branch of the tree ends with a white vertex. If we recall that Γ is obtained from Γ_{para} by adding a single "special" vertex, then we find that our graph Γ_1 has distinguished vertices: either one of them, v_1, or two of them, v and v_1—these are the neighbors of the "special" vertex.

Consider now an arbitrary vertex w of Γ_1. Since Γ_1 is a tree, there exists a unique chain joining v and w. Blow down all the remaining vertices not belonging to this chain; from our description of Γ_1 we have given it is easy to see that this can be done. Then we conclude that there exists a weighted graph of the form given in Figure 1(a) or (b) that is a predecessor of Γ.

Suppose that we delete the vertex w from this graph; by definition of a Lannér graph, what is left is either elliptic or parabolic. For this graph we introduce new multiplicities, in terms of the fundamental cycle. Then the new multiplicity of the white vertex v will be $t-1$. It follows from this that Γ contains an elliptic or parabolic subgraph, and a white vertex in it of multiplicity $\geq t-1$. Then by Corollary 3.9 and Lemma 3.8(i), (ii), we have $(t-1)\varepsilon \leq 2$, that is, $t \leq 1 + 2/\varepsilon$.

We return to the graph Γ_1; in it, the black vertices form a connected subgraph, which by Corollary 3.11 is of the form A_n, D_n or E_n. The length of the chain from any vertex to a distinguished vertex is at most $1 + 2/\varepsilon$. Hence the number of black vertices of Γ_1 is at most $4/\varepsilon$.

We now consider the sum $A = \sum a_j$ of the multiplicities of the white vertices, and observe how A and the number of vertices changes in Steps 1, 2 and 3.

Proof of Theorem 3.13, *continued.* In Step 1, the number of white vertices is exactly equal to A. In Step 2, the number of vertices increases by at most 6. Consider what happens to the sum A. Let w be the vertex appearing on blowing up the edge (v_1, v_2); the multiplicity of w equals the sum of the multiplicities of v_1 and v_2. If both of these are white then A remains unchanged; but this situation can occur once only, since in this case the graph being blown up consists just of v_1 and v_2. If at least one of v_1 and v_2 is black then the sum increases by at least 1. Thus in this case, the number of vertices increases at most 6 times as fast as A. At Step 3, A obviously increases faster than the number of vertices.

Now consider the final graph Γ_{para}. By Lemma 3.8(i), we have $A \cdot \min(\eta_j) \leq 2$, and since Γ_{para} satisfies $DP(\varepsilon)$, it follows that $A \leq 2/\varepsilon$.

Taking the sum of all that was said above, we get the bound $1 + 4/\varepsilon + 6 + 6 \cdot 2/\varepsilon = 7 + 16/\varepsilon$ for the number of vertices of Γ. To finish the proof in case (iii), it remains only to notice that a vertex not belonging to O cannot be blown up, since this violates the condition that the graph is Lannér.

Case (iv). Take, for example, the graph $G(1,2;2)$. After successive blow-ups, the possibilities are Figure 1(c) or (d) . (c) is the case when a cycle occurs, made up of white or black vertices indiscriminately. In this case we assert that only edges are blown up. Indeed, suppose that a vertex is blown up. If the internal cycle is hyperbolic then the graph we obtain is not Lannér; otherwise, we get a contradiction from Lemma 3.8(i) and (ii), as in Corollary 3.10.

In $G(1,2;2)$, consider the cycle $C_0 = v_1 + v_2$. By Lemma 3.8(iii), the sum A of multiplicities in the cycle $C = \sigma^* C_0$ satisfies $A \cdot \varepsilon \leq 3$. Since only edges are blown up, the sum A grows faster than the number of vertices, and we get the bound $2 + 3/\varepsilon < 7 + 16/\varepsilon$ for the number of vertices of Γ.

In the case of Figure 1(d), only one blow-up is allowed, and this leads to the graph (e). If a vertex is blown up following a sequence of blow-ups which does not pass through the graph (e) then we use the same argument as in (c). Here we use the fact that a parabolic or elliptic graph that is a tree with a fork, having a white vertex at the fork, always has arithmetic genus greater than zero. If the sequence of blow-ups passes through (e) then there cannot be any blow-ups of vertices, for otherwise there would exist a predecessor graph with a black subgraph of type $\widetilde{\mathrm{D}}_n$. Furthermore, as in case (c), we get the bound $4 + 3/\varepsilon < 7 + 16/\varepsilon$.

The graph $G(1,3;2)$ is treated in the same way, and we get the same bound.

THEOREM 3.15. *Suppose that a weighted graph $\widetilde{\Gamma}$ satisfies $DP(\varepsilon)$ with $\varepsilon > 0$. Then for any connected elliptic subgraph $\Gamma \subset \widetilde{\Gamma}$ we have the bounds $c_1 \leq 16 + 44/\varepsilon$ and $c_2 \leq 17 + 44/\varepsilon$ for the constants c_1 and c_2 of Theorem* 2.2.

The proof of this theorem is similar to that of Theorem 3.13. We note again that by Corollary 3.9, the graph Γ also satisfies $DP(\varepsilon)$. In [7] it is proved that Γ is log terminal. In our treatment, this follows immediately from Lemma 3.7(i) and (ii). Now in Γ we contract all the white vertices; we obtain a morphism $\sigma \colon \Gamma \to \Gamma_0$ where Γ_0 is the graph of a minimal resolution of a log terminal singularity. These graphs are listed in [5] and, as already mentioned, are weighted graphs of type A_n, D_n or E_n.

LEMMA 3.16. *The morphism $\sigma \colon \Gamma \to \Gamma_0$ can be written as a composite of blow-ups as follows: Step* 1, *then repeated use of Step* 2A, *then repeated use of Step* 2B *and Step* 3.

Step 1. *We blow up only vertices.*

Step 2A. *We first blow up an edge (v_1, v_2) where v_1 and v_2 are vertices of multiplicity* 1, *then blow up the newly appeared vertex (compulsory), then optionally the newly appeared vertex, then again optionally the most recently appeared vertex, and so on.*

Step 2B. *We blow up an edge (v_1, v_2). Either both v_1 and v_2 have multiplicity* 1 *(and then nothing more takes place), or we have a sequence of blow-ups as in Step* 2A. *The number of vertices appearing in this step is at most* 6.

Step 3. *We blow up only black vertices which appeared in Step* 2.

The proof of the lemma is similar to that of Lemma 3.14. We note only that if after blowing up an edge we contract white vertices in the left-hand and right-hand graphs to obtain graphs consisting of only one vertex, then v_1 and v_2 have multiplicity 1.

Proof of Theorem 3.15, *continued.* The essential remark is that after Step 1 and Step 2A, the black vertices form a connected subgraph which by Corollary 3.11 is a graph of type A_n, D_n or E_n. In this black subgraph, the number of vertices at distance $d \le l$ from a fixed vertex is at most $2l+2$. Obviously, c_1 is bounded by $2l+2$ plus the number of remaining vertices introduced in Steps 1, 2A, 2B and 3. As in the proof of Theorem 3.13, this number is bounded by $12/\varepsilon$. Thus using (3.13), we get the bound $c_1 \le 2(7+16/\varepsilon)+2+12/\varepsilon = 16+44/\varepsilon$. Similarly, $c_2 \le 17+44/\varepsilon$.

§4. Fractional indices

Let X be a log del Pezzo surface and $r(X)$ its fractional index. If X is obtained by contracting the negative section on the rational scroll $\mathbb{F}_n$ then $r = 1+2/n$; in particular, $r(\mathbb{P}^2) = 3$. In all other cases, it follows from Proposition 1.7 that $r \le 1$.

Consider the set R given by

$$R = \{r(X) \mid X \text{ is a log del Pezzo surface}\} \subset [0,3].$$

In this section we describe the accumulation points of R and their nature.

LEMMA 4.1. *Let $\sigma\colon Y \to M$ be a birational morphism of a smooth surface Y to $\mathbb{P}^2$ or $\mathbb{F}_n$; suppose that $\rho(Y)-\rho(M) = k$. Then for fixed M and k, the selfintersection of an exceptional curve C of Y is bounded, that is,* $0 > C^2 \ge \text{const}$.

PROOF. The blow-up of $\mathbb{P}^2$ at one point is $\mathbb{F}_1$, so that we consider at once the general case. We have $\sigma\colon Y \to \mathbb{F}_n$; let C be an exceptional curve of Y, that is, an irreducible curve with $C^2 < 0$. Then $\sigma_* C = D$ is an irreducible curve of $\mathbb{F}_n$, and C is obtained from D by blowing up several times in points of multiplicity $m_1, \dots, m_{k'}$, where $k' \le k$ and $m_i \ge 1$. Thus $C^2 = D^2 - \sum m_i^2$. The two natural conditions $-K_Y C \le 1$ and $2p_a C \ge 0$ can be rewritten in the form

$$\sum m_i \ge -KD-1 \quad \text{and} \quad \sum (m_i^2 - m_i) \le D^2 + KD + 2,$$

where $K = K_{\mathbb{F}_n}$. Obviously, $\left(\sum(m_i-1)\right)^2 \ge \sum(m_i-1)^2$ can be rewritten as

$$D^2 + c_1 DK + c_2 \ge (DK)^2,$$

where c_1 and c_2 are certain constants. We assume that D does not coincide with the negative section of $\mathbb{F}_n$, so that $D^2 \ge 0$. Since the lattice $\operatorname{Pic}\mathbb{F}_n$ is hyperbolic, we have

$$D^2 \le \frac{(DK)^2}{K^2} = \frac{1}{8}(DK)^2.$$

Thus $c_1 DK + c_2 \ge \frac{7}{8}(DK)^2$ and $-DK \le c_3$. There are obviously only a finite set of numerically nonequivalent curves D with $D^2 \ge 0$ satisfying this condition. Hence $\sum m_i^2$ is bounded, and with it $-C^2$.

LEMMA 4.2. *Let $\sigma\colon Y \to \mathbb{F}_n$ be a birational morphism of smooth surfaces with $n \ge k = \rho(Y) - \rho(\mathbb{F}_n)$. Then, except for the proper transform of the negative section, all the exceptional curves of Y are contained in fibers of the morphism $Y \to \mathbb{F}_n \to \mathbb{P}^1$.*

PROOF. In fact, let C be another exceptional curve on Y with $D = \sigma_* C \sim a s_n + bf \ne s_n$, where s_n and f are the negative section and fiber of $\mathbb{F}_n$. Then $D^2 \ge a^2 n$ and all $m_i \le a$ (since D is an irreducible curve and $D \cdot s_n \ge 0$). Hence $C^2 = D^2 - \sum m_i^2 \ge a^2 n - a^2 k \ge 0$.

THEOREM 4.3. *The set R has the following accumulation points:* 0 *and* $1/m$ *for any natural number* m; *all of these points are limit points from above and not from below.*

PROOF. We show first that there are no other accumulation points in R. Fix some $\varepsilon > 0$ and consider the accumulation points of the set $R \cap [\varepsilon, +\infty[$. If $r(X) \geq \varepsilon$, then by Theorem 2.4 we have a bound $\rho(Y) \leq F(\varepsilon)$ for the rank $\rho(Y)$ of the Picard lattice of the minimal resolution of singularities of X. There exists a birational morphism $\sigma: Y \to M$ from Y to a nonsingular minimal model. If we are only interested in the limit points of R, we can assume that $n \geq F(\varepsilon)$; indeed, for any fixed surface M, the selfintersection number C_i^2 of the exceptional curves C_i is bounded by Lemma 4.1. As we know from Proposition 1.3, the exceptional curves F_i on Y with $F_i^2 \leq -2$ form a log terminal graph. There are only finitely many such graphs, since we have bounded the number of vertices and the weight of each vertex; thus there are only finitely many possibilities for the DP-coefficients $\eta_j = 1 + \sum \alpha_i F_i \cdot F_j$ and therefore only finitely many possible fractional indices $r = \gcd(\eta_j)$.

Suppose then that $n \geq F(\varepsilon)$. Then from Lemma 4.2 we deduce a description of the graph of exceptional curves on Y. We get a finite number of series of weighted graphs; in each series, the graphs differ only in a natural number n, the weight of one of the vertices. It is enough to keep track of how r changes in one such series; consider Γ_n for $n = F(\varepsilon), F(\varepsilon) + 1, \ldots$. Let F_0 be the inverse image of the negative section, $F_0^2 = -n$. We divide up all the remaining curves into sets

$$\{F_i, E_j\}_{i \in I_1, j \in J_1}, \ldots, \{F_i, E_j\}_{i \in I_s, j \in J_s},$$

according as to which fiber of $Y \to \mathbb{F}_n \to \mathbb{P}^1$ they belong to; here $E_j^2 = -1$ and $F_i^2 \leq -2$. Write f for one fiber of this morphism. We have the numerical equality

$$f = \sum_{j \in J_1} a_j E_j + \sum_{i \in I_1} b_i F_i = \cdots = \sum_{j \in J_s} a_j E_j + \sum_{i \in I_s} b_i F_i.$$

We have

$$-\pi^* K_X \cdot f = \left(-K_Y + \sum \alpha_i F_i\right) \cdot f = 2 + \alpha_0.$$

On the other hand,

$$-\pi^* K_X \cdot f = \sum_{j \in J_1} a_j \eta_j = \cdots = \sum_{j \in J_s} a_j \eta_j.$$

Let $\Delta = \Delta(n) = A_n + B$ be the determinant of the connected black subgraph of Γ_n containing F_0. Then all η_j are of the form $\eta_j = \nu_j(n)/\Delta(n)$, where the $\nu_j(n)$ are linear functions with rational coefficients; $r = \gcd(\eta_j) = \gcd(\eta_j(n))/\Delta(n)$. If not all of the functions $\nu_j(n)$ are linearly proportional then $r = \gcd(a_1, b_1 n + b_2)/\Delta(n) \to 0$. Now suppose that all the $\nu_j(n)$ are proportional. We write $\overline{\eta_j}$ and $\overline{r}$ for the limiting values. Then $\overline{r} = \overline{\gcd(\eta_j)} = \gcd(\overline{\eta_j})$. We have

$$\sum_{j \in J_1} a_j \overline{\eta_j} = \cdots = \sum_{j \in J_s} a_j \overline{\eta_j} = 2 + \lim_{n \to \infty} \alpha_0(n).$$

LEMMA 4.4. *Let* Γ_n *be a series of elliptic graphs with variable weight at one vertex* F_0, *and* α_i *the coefficients defined in Definition* 3.1. *Then* $\lim_{n \to \infty} \alpha_0(n) = -1$; *moreover, if* $\alpha_0(n_0) > -1$ *for some* n_0 *then* $\alpha_0(n)$ *is a decreasing function, and if* $\alpha_0(n_0) < -1$ *then it is an increasing function.*

PROOF. From the system of linear equations $\sum \alpha_i F_i \cdot F_j = -F_j^2 - 2$, we get

$$\sum (1+\alpha_i) F_i \cdot F_j = q_j - 2, \qquad \text{where } q_j = \sum_{i \neq j} F_i \cdot F_j.$$

From this system we get $\alpha_0 = -1 + \text{const}/\Delta(n)$, and the assertion of the lemma follows from this.

Thus

$$\sum_{j \in J_1} a_j \overline{\eta_j} = \cdots = \sum_{j \in J_s} a_j \overline{\eta_j} = 1.$$

Let $\mu_k = \gcd\{\overline{\eta_j} \mid j \in J_k\}$ and $\overline{\eta_j} = t_{kj} \cdot \mu_k$. Then $\mu_k \sum a_j t_{kj} = 1$, where $s_k = \sum a_j t_{kj}$ is an integer, and $\mu_k = 1/s_k$. Finally,

$$r = \gcd\{1/s_k \mid k = 1, \ldots, s\} = 1/\operatorname{lcm}(s_k).$$

Thus all the accumulation points of R are of the form $1/m$, and by Lemma 4.4 they are limit points from above.

We now show that all points of the form $1/m$ really are accumulation points. For $m = 1$ we have already proved this. Corresponding series for $m \geq 2$ are given in the following example.

EXAMPLE 4.5. Blow up one point on $\mathbb{F}_n$, then the point of intersection of the two components of the resulting fiber. We get the graph of exceptional curves shown in Figure 2(a). We now repeat the following operation a number of times: blow up a point on the surface corresponding to the edge connecting the white vertex to a black one—the left-hand or right-hand one, at will. We get a surface Y containing in particular the exceptional curves corresponding to the graph of Figure 2(b), in which the shaded boxes represent chains of black vertices. There is a standard way of associating with each such chain a reduced fraction $0 < a/d < 1$ (see [4]). It is easy to see that every such fraction can be obtained as described. Now contract on Y the curves corresponding to black vertices; this is possible by [3]. We get a normal projective surface X with log terminal singularities and $\operatorname{Pic} X = \mathbb{Z}$. This is a del Pezzo surface, since the curve E corresponding to the white vertex satisfies

$$-\pi^* K_X \cdot E = \frac{1}{d}\left(1 + \frac{d+1}{nd-a}\right) > 0.$$

In this series, $r = -\pi^* K_X \cdot E$ tends to $1/d$. This completes the proof of Theorem 4.3.

Consider the particular case $a = d-1$, $n = 2$. In this case $\eta = 2/d$, $r = 2/d$, $k = d$ or $d/2$, and $\rho(Y) = d+1$. Thus the bounds in Theorems 2.3, 2.4 and 2.5 cannot be made better than linear in $1/\varepsilon$, $1/r$ and k respectively.

EXAMPLE 4.6. If we consider del Pezzo surfaces not with log terminal singularities, but with a wider class of singularities, for example rational singularities, then fractional indices can accumulate from below. In Example 4.5, we performed blow-ups in one fiber of $\mathbb{F}_n$; if however, we blow up 5 fibers of $\mathbb{F}_n$ in the same way, we can obtain a surface with the graph of exceptional curves shown in Figure 2(c).

If we now contract the curves corresponding to black vertices, we get a series of surfaces X_n with $\operatorname{Pic} X_n = \mathbb{Z}$ in which the indices tend to $1/m$ from below.

If Theorem 2.4 holds for the chosen class of singularities, then a word-for-word repetition of the proof of Theorem 4.3 shows that there are no accumulation points other than $\{0, 1/m\}$.

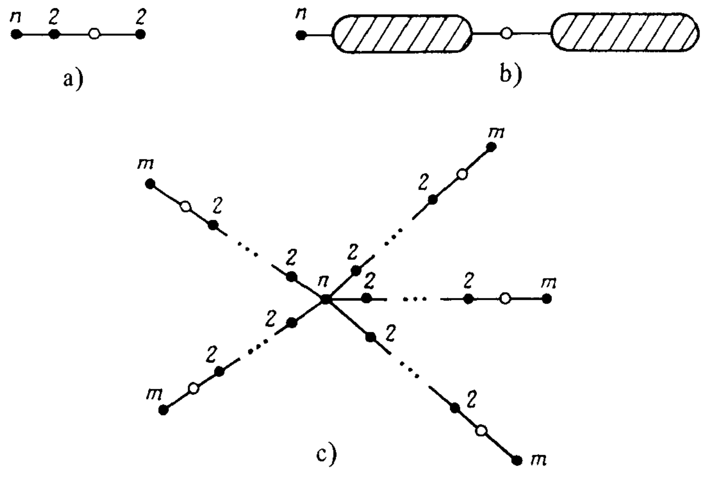


FIGURE 2, (a), (b), (c)

Theorem 4.3 and Example 4.6 give food for certain reflections. If we consider the absence of limiting from below as "good" behavior, we see that log del Pezzo surfaces form a natural class of singular surfaces, rather than an arbitrarily selected class.(2)

We also note that for every natural m, the proof of Theorem 4.3 actually allows us to describe all possible graphs of exceptional curves on a log del Pezzo surface with $0 < |r(X) - 1/m| < \varepsilon_m$ for sufficiently small ε_m.

For explicit computations of fractional indices, we need explicit formulas for the coefficients α_i. We give such formulas below for singularities corresponding to a graph having at most one fork; these include the graphs A_n, D_n, and E_n, the only graphs appearing in the log terminal case.

LEMMA 4.7. *Let Γ be a weighted graph with simple edges, v one vertex of Γ, having weight p_v, and $v_1, \dots, v_k$ the neighboring vertices. Then*

$$\Delta(\Gamma) = -p_v \cdot \Delta(\Gamma - v) - \sum_i \Delta(\Gamma - v - v_i),$$

where Δ is the absolute value of the determinant of the corresponding quadratic form.

The proof is obvious.

PROPOSITION 4.8. *The coefficients α_i of a graph Γ can be calculated as follows.*

(i) *Suppose that Γ is a chain, and that the vertex v breaks Γ up into graphs Γ_1 and Γ_2; let Δ, Δ_1 and Δ_2 be the absolute values of the determinants of the corresponding quadratic forms. Then*

$$\alpha(v) = -1 + \frac{\Delta_1 + \Delta_2}{\Delta}$$

(*here, if v is an end vertex and Γ_1, say, is empty, then $\Delta_1 = 1$*).

(ii) *Suppose that Γ is a tree with a single fork not at v, and that v breaks Γ up into a chain Γ_1 and a graph Γ_2; write Γ_A and Γ_B for the connected components of the graph*

(2) *Translator's note.* This view is unnecessarily defensive: by work of Iitaka and Kawamata, the log category is now firmly established on an equal footing with the usual category of classification theory. As Shokurov points out, the appropriate reflection here is that we should expect similar limiting behavior for other phenomena in classification of varieties with $-K$ ample.

$G \setminus \{fork\}$ not containing v. Let Δ, Δ_1, Δ_2, Δ_A and Δ_B be the absolute values of the determinants of the corresponding quadratic forms. Then

$$\alpha(v) = -1 + \frac{\Delta_1 + \Delta_2 - \Delta_1(\Delta_A - 1)(\Delta_B - 1)}{\Delta}.$$

(iii) *Suppose that Γ is a tree with one fork at v, and that v breaks Γ up into graphs Γ_1, Γ_2 and Γ_3. Then*

$$\alpha(v) = -1 + \frac{\Delta_1\Delta_2\Delta_3}{\Delta}\left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \frac{1}{\Delta_3} - 1\right).$$

PROOF. We substitute the α_i into the linear equations $\sum \alpha_i F_i \cdot F_j = -F_j^2 - 2$ and, using Lemma 4.7, verify that they are satisfied identically.

I proved in [1] that on a log del Pezzo surface X of index 1 or 2, the linear system $|-2K_X|$ contains a smooth element; the result of this paper allows us to generalize this somewhat.

PROPOSITION 4.9. *Let X be a log del Pezzo surface of index k and fractional index r. Then for $N \geq \rho(Y)/2r$, the linear system $|-NkK_X|$ is nonempty, has no fixed part, and contains a smooth element.*

In particular, by Theorem 2.5, there exists an absolute constant A such that all the above holds for the linear system $|-Ak^3K_X|$.

PROOF. Let $\pi\colon Y \to X$ be a minimal resolution of singularities. It is enough to proof all the assertions for the linear system $|-N\pi^*(kK_X)|$ on Y. The whole proof proceeds in complete analogy with that of [1], Theorem 1.4.1, except for one point: to prove that there are no fixed components, one has to prove the following inequality. Let E be the fixed part of $|-N\pi^*(kK_X)|$. Then

$$\left(-2N\pi^*(kK_X) - K_Y - E\right) \cdot E \neq 0.$$

For $\beta \leq NK$, write $E = \beta(-\pi^*K_X) + F$ with $F \in (\pi^*K_X)^\perp$ in $\operatorname{Pic} Y \otimes \mathbb{Q}$. Then

$$\begin{aligned}\left(-2N\pi^*(kK_X) - K_Y - E\right) \cdot E &= (2Nk + 1 - \beta)\beta K_X^2 - \left(\sum \alpha_i F_i + F\right) \cdot F \\ &> Nk \cdot \beta K_X^2 - \left(\sum \alpha_i F_i + F\right) \cdot F.\end{aligned}$$

The first term in this sum is $\geq Nkr$, since $\frac{1}{r}\beta K_X^2 = E \cdot (-\pi^*K_X/r)$ is an integer. The second term takes a minimum value $\frac{1}{4}\left(\sum \alpha_i F_i\right)^2$ when $F = -\frac{1}{2}\sum \alpha_i F_i^2$. But

$$-\left(\sum \alpha_i F_i\right)^2 = -\sum \alpha_i(-F_i^2 - 2) \leq 2k\rho(Y),$$

since $-\alpha_i \leq 1$ and $F_i^2 \leq 2k$. Thus it is sufficient that the inequality $Nkr - \frac{1}{2}k\rho(Y) \geq 0$ holds; that is, $N \geq \rho(Y)/2r$.

Moscow State University Received 15/MAR/88

REFERENCES

1. V. A. Alekseev and V. V. Nikulin, *Classification of del Pezzo surfaces with log terminal singularities of index ≤ 2 and involutions on $K3$ surfaces*, Dokl. Akad. Nauk SSSR **306** 1(989), 525–528; English transl. in Soviet Math. Dokl. **39** (1989). See also Voprosy Chistoĭ i Prikl. Mat. (Dokl. po Mat. i Prilozhen.) **2** (1988), no. 2 (to appear). (Russian).

2. Michael Artin, *Some numerical criteria for contractibility of curves on algebraic surfaces*, Amer. J. Math. **84** (1962), 485-496.

3. ——, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129-136.

4. Egbert Brieskorn, *Rationale Singularitäten komplexer Flächen*, Invent. Math. **4** (1967/68), 336–358.

5. A. I. Iliev, *Log terminal singularities of algebraic surfaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1986**, no. 3, 38–44; English transl. in Moscow Univ. Mech. Bull. **41** (1986).

6. V. V. Nikulin, *Del Pezzo surfaces with log terminal singularities*. I, Mat. Sb. **180** (1989), no. 2, 226–243; English transl. in Math. USSR Sb. **66** (1990).

7. V. V. Nikulin, *Del Pezzo surfaces with log terminal singularities*. II, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), 1032-1050; English transl. in Math. USSR Izv. **33** (1990).

8*. M. Reid, *Campedelli versus Godeaux*, Preprint, Res. Inst. Math. Sci., Kyoto, 1989: submitted to Proc. CIME Conf. on Algebraic Surfaces (Cortona, 1988; F. Catanese, et al., editors), CIME (to appear).

Translated by M. REID

---

* *Added in translation.*