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AMPLE WEIL DIVISORS ON K3 SURFACES WITH DU VAL SINGULARITIES

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0.1. The motivation of this paper comes from the study of **Q**-Fano varieties. **Q**-Fanos are one of the classes of (singular) varieties that naturally appear in the Minimal Model Program; see [KMM], [Ko], [Mr], or [W] for the introduction.

0.2. Nonsingular Fano varieties (i.e., varieties with the ample anticanonical class $-K_X$) with Picard number $\rho(X) = 1$ were classified by G. Fano and V. A. Iskovskikh; see [I1], [I2]. Among the first steps in this classification are the following.

(i) Find a smooth surface S in the anticanonical linear system $|-K_X|$ (done in [Sh]). By adjunction formula and Kodaira vanishing, it is a K3 surface.
(ii) Restrict $|-K_X|$ on S. It is an ample linear system. Now use the following to obtain the description of $|-K_X|$.

THEOREM 0.2.1 ([SD]). *Let $|D|$ be an ample complete system on a smooth* K3 *surface. Then $|D|$ is either free or has the single base component C of multiplicity one and $|D| = C + |nE|$ where $|E|$ is an elliptic pencil.*

0.3. For the singular **Q**-Fanos one can try to use the same approach. The first difference is that, if X has a non-Gorenstein singularity, then locally in a neighborhood of such a point a general element of $|-K_X|$ should have Du Val singularity. The second observation is that $-K_X$, restricted on S, is not a Cartier divisor any more but only a Weil divisor such that its multiple is an ample Cartier divisor.

Therefore, we see that in order to work with the singular case we have to consider a $K3$ surface with Du Val singularities and an ample Weil divisor on it.

0.4. Some results in the direction of 0.2.(i) are contained in [A]. Among them is the following theorem.

THEOREM 0.4.1. *Let X be a* **Q**-*Fano of degree $d = (-K_X)^3 \geqslant 4$. Then one of the following is true.*

(i) *A general element $S \in |-K_X|$ has (not worse than) Du Val singularities;*
(ii) *X is birationally equivalent to another* **Q**-*Fano X_1 such that a general element $S \in |-K_{X_1}|$ has Du Val singularities;*

0.5. This paper gives some answers for the second half of 0.2. Here we study the

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question when $|D|$ can have multiple base curves. The main result is the following theorem.

THEOREM 2.3. *Let D be an ample Weil divisor on a* K3 *surface with Du Val singularities and $\rho(S) = 1$. Assume that one of the following is true.*

(i) $D^2 > 12\frac{25}{42}$;
(ii) $h^0(D) > 7$.

Then the linear system $|D|$ does not have multiple base curves.

Both inequalities in (i) and (ii) are sharp.

Remark 0.6. A general K3 surface in $|-K_X|$ on a nonsingular Fano variety X with Picard number one also has Picard number one (by [Msh] and classificational results [I1], [I2]). So the condition $\rho(S) = 1$ does not seem to be unnatural.

In the case $\rho(S) > 1$ there are examples due to V. V. Nikulin [N] when D^2 and $h^0(D)$ are arbitrarily large, but $|D|$ has base curves of multiplicity up to 5.

Remark 0.7. Theorem 2.3 might also help for 0.2.(i). Combined with other results, it can lead to a stronger statement than 0.4.1, possibly with other conditions on the degree of X.

Remark 0.8. Lemma 1.8. below and part (i) of Theorem 2.3 with the estimate $D^2 > 20$ were proved independently by V. V. Nikulin in [N]. The method of [N] is to consider the base components of a linear system on the minimal desingularization $\tilde{S}$ of S, i.e. a usual smooth K3 surface; the method involves very hard computations with weighted graphs. See also [U].

1. Fixed curves on S

1.1. We restrict ourselves to the following situation.

(i) S is a K3 surface with Du Val singularities. In particular, $D_S(K_S) \simeq O_S$ and $h^1(O_S) = 0$.
(ii) $\rho(S) = 1$. In particular, any nonzero effective divisor D on S is ample.

We use the following two basic results.

FORMULA 1.2. (Riemann-Roch formula for surfaces with Du Val singularities; [R, 9.1]).

$$\chi(O_S(D)) = \chi(O_S) + \frac{D(D - K_S)}{2} - \sum_Q \frac{i(r-i)}{2}.$$

Here the summation goes over singular points. For a cyclic Du Val point (i.e., a point of type A_n), one has $r = n + 1$, and the number $0 \leqslant i < r - 1$ describes the type of the divisor D at Q.

Every noncyclic point (i.e., of type D_n or E_n) corresponds to the "basket" of cyclic points. For example, D_4 splits into 4 points of type A_1. It is essential that the correspondence between divisors and coefficients $i \in Z_r$ is a homomorphism of groups.

THEOREM 1.3. (vanishing theorem of Kawamata and Viehweg; [KMM, 1-2-5]). *If a divisor* $D - K_X$ *is ample, then* $H^i(S, O_X(D)) = 0$ *for* $i > 0$.

LEMMA 1.4. *Let S be a* K3 *surface with Du Val singularities and C be an effective curve on S. Suppose that one of the following conditions is true.*

(i) *C is connected and reduced, or*
(ii) *rk* Pic $S = 1$.

Then $h^0(O_S(C)) = p_a(C) + 1$.

COROLLARY 1.5. *For a connected and reduced curve C* $h^0(O_S(C)) = 1$ *if and only if C is a tree of nonsingular rational curves.*

Proof of Lemma 1.4. It is almost the same as for the nonsingular case. One has the exact sequence

$$0 \to I \to O_S \to O_C \to 0$$

where I is the ideal sheaf of the curve C. By the definition it is the ideal of functions vanishing on C, i.e., $O_S(-C)$ and

$$\begin{aligned}\chi(O_S(C)) &= \chi_S(O_S(-C)) \\ &= \chi(O_S) - \chi(O_C) \\ &= 2 - (1 - p_a(C)) = 1 + p_a(C).\end{aligned}$$

A little secret of these formulas that look almost like a tautology is that, for a function $f(i) = i(r-i)/2r$, one has $f(i) = f(r-i)$.

Now by duality $h^2(O_S(C)) = h^0(O_S(-C)) = 0$. If *rk* Pic$(S) = 1$, then $h^1(O_S(C)) = 0$ by the vanishing theorem, Theorem 1.3. In case (i), one has $h^0(O_C) = 1$, and by the same exact sequence

$$h^1(O_S(C)) = h^1(O_S(-C)) = h^0(O_C) - h^0(O_S) = 0.$$

Therefore,

$$h^0(O_S(C)) = \chi(O_S(C)) = p_a(C) + 1. \qquad \square$$

From now on we suppose that $\rho(S) = 1$.

1.6. For purposes that will be explained below, let us introduce the functions

$$f(k) = \frac{k(r-k)}{2r},$$

$$\bar{f}(k) = \frac{\bar{k}(r-\bar{k})}{2r}, \qquad \text{and}$$

$$h(k) = \bar{f}(k) - f(k),$$

where $\bar{k}$ is the residue of k modulo r. One can easily see that $f(k) - f(k+r) = k$. Therefore, if $0 \leqslant k < r$, then

$$h(k) = 0,$$

$$h(k+r) = k,$$

$$h(k+2r) = k + (k+r), \qquad \text{etc.}$$

1.7. Let C be a smooth rational curve on S. Suppose that C passes through several singular points and $(r_1, i_1)\dots(r_s, i_s)$ is the corresponding "basket" of cyclic singularities as in Formula 1.2. Symbols (r, i) and $(r, r-i)$ denote the same singularities up to choosing another generator in the group $\mathbf{Z}_r$. Hence, we can always assume that $2i \leqslant r$.

By Corollary 1.5, $h^0(C) = 1$. The Riemann-Roch formula, 1.2, and the vanishing theorem, 1.3, give

$$2 + \frac{C^2}{2} - \sum_Q f(i) = 1. \tag{1}$$

Similarly, if $h^0(nC) = 1$, $n > 1$, then

$$2 + n^2\frac{C^2}{2} - \sum_Q \bar{f}(ni) = 1. \tag{n}$$

LEMMA 1.8. (i) $h^0(2C) = 1$ *if and only if*
 (a) $s = 3$ *and* $i_1 = i_2 = i_3 = 1$,
 (b) $s = 2$ *and* $i_1 = 1, i_2 = 2$,
 (c) $s = 1$ *and* $i_1 = 3$;
(ii) $h^0(3C) = 1$ *if and only if*
 in case (a), $r_1 = 2, r_2 \geqslant 3$,
 in case (b), $r_1 = 2, r_2 \geqslant 6$;
(iii) $h^0(4C) = 1$ *if and only if*
 in case (a), $r_1 = 2, r_2 = 3, r_3 \geqslant 4$;
(iv) $h^0(5C) = 1$ *if and only if* $h^0(4C) = 1$;
(v) $h^0(6C) > 1$ *for any case.*

Proof. Eliminating C^2 from (1) and (n), we see that the formula (n) is equivalent to

$$\sum \frac{n(n-1)}{2} i = (n^2 - 1) + \sum h(ni). \tag{n^*}$$

With our assumptions (see 1.7), $h(2i) = 0$; so the condition $h^0(2C) = 1$ is equivalent to

$$\sum i = 3,$$

and we get (i). It is an easy exercise to get (ii)–(v) from (3*)–(6*). □

1.9. Note that, since $\rho(S) = 1$, $C^2 > 0$; so, from (1), one has

$$\sum \frac{i(r-i)}{r} < 2.$$

This is an additional condition on r_1, r_2, r_3. For example, in case 1.8(i)(a),

$$\sum \frac{1}{r} < 1.$$

2. Base curves in linear systems

2.1. Let $|D|$ be an arbitrary nonempty system on S. Suppose that $|D|$ has a base component C (which is a smooth rational curve) of multiplicity at least 2; so $h^0(D) = h^0(D - C) = h^0(D - 2C)$. Let us denote by $0 \leqslant m_1 < r_1 \ldots 0 \leqslant m_s < r_s$ invariants corresponding to the "basket" of cyclic singularities and the divisor $D - 2C$, as in 1.2. Then by the Riemann-Roch formula, 1.2, and the vanishing theorem, 1.3,

$$h^0(D) = 2 + \frac{D^2}{2} - \sum \bar{f}(m + 2i), \tag{d}$$

$$h^0(D - C) = 2 + \frac{(D - C)^2}{2} - \sum \bar{f}(m + i), \tag{d1}$$

$$h^0(D - 2C) = 2 + \frac{(D - 2C)^2}{2} - \sum \bar{f}(m). \tag{d2}$$

From equalities (d), (d1), and (1), one can find that

$$DC = -1 + \sum \bar{f}(m + 2i) + \bar{f}(i) - \bar{f}(m + i). \tag{dc}$$

Similarly, from (d), (d2), and (2),

$$2DC = -1 + \sum \bar{f}(m + 2i) + \bar{f}(2i) - \bar{f}(m). \tag{2dc}$$

Eliminating DC in (dc) and (2dc), one gets

$$\sum \bar{f}(m+2i) + \bar{f}(m) + 2\bar{f}(i) - 2\bar{f}(m+i) - \bar{f}(2i) = 1.$$

It is easy to verify that for the function f (not $\bar{f}$) one has $f(m+2i) + f(m) + 2f(i) - 2f(m+i) - f(2i) = 0$.

Therefore, with our assumptions on i and m we get the final equality

$$\sum_{Q} h(m+2i) - 2h(m+i) = 1. \qquad (**)$$

THEOREM 2.2. *Suppose that C is a base curve in $|D|$ of multiplicity at least 2. Then in the previous notations (see Lemma 1.8) one has*

in case (a), $m_1 = r_1 - 1, m_2 \leqslant r_2 - 2, m_3 \leqslant r_3 - 2$;
in case (b), *either* $m_1 = r - 1, m_2 \leqslant r - 4$,
or $m_1 \leqslant r_1 - 2, m_2 = r_2 - 3$,
or $m_1 = r_1 - 2, m_2 = r_2 - 1$;
in case (c), *either* $m_1 = r_1 - 5$ *or* $m_1 = r_1 - 1$.

Proof. By the definition of function $h(k)$, one has

$$h(m+2i) = \begin{cases} 0 & \text{if } m+2i \leqslant r \\ m+2i-r & \text{if } m+2i \geqslant r \end{cases}$$

and

$$h(m+i) = \begin{cases} 0 & \text{if } m+i \leqslant r \\ m+i-r & \text{if } m+i \geqslant r. \end{cases}$$

So

$$h(m+2i) - 2h(m+i) = \begin{cases} 0 & \text{if } m+2i \leqslant r \\ m+2i-r & \text{if } m+2i \geqslant r \text{ and } m+i \leqslant r \\ r-m & \text{if } m+2i \geqslant r \text{ and } m+i \geqslant r. \end{cases}$$

Finally, (**) easily implies the statement. □

Now we are ready to prove the main theorem.

THEOREM 2.3. *Let D be an ample Weil divisor on a K3 surface S with Du Val singularities and $\rho(S) = 1$. Assume that one of the following is true.*

(i) $D^2 > 12\frac{25}{42}$;
(ii) $h^0(D) > 7$.

Then the linear system $|D|$ does not have multiple base curves.

Proof. Suppose that we are in case (a) of 1.8 and 2.2. So, we have 3 cyclic singularities and $i_1 = i_2 = i_3$. From (1) one has

$$C^2 = 1 - \sum \frac{1}{r}.$$

From this equality and (dc), one has the expression for D^2

$$D^2 = \frac{\left(\sum \bar{f}(m+2) - \bar{f}(m+1) + \frac{1}{2}\left(1 - \sum \frac{1}{r}\right)\right)^2}{1 - \sum \frac{1}{r}}.$$

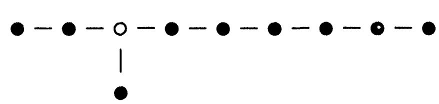
It is easy to see that $\bar{f}(m+2) - \bar{f}(m+1)$ has the maximal value when $m = r - 1$, and the next value for $m = 0$. Therefore, in the conditions of Theorem 2.2 the maximum of D^2 is achieved for $m = r_1 - 1$, $m_2 = m_3 = 0$ and is equal to

$$D^2 = \frac{\left(2\left(1 - \sum \frac{1}{r}\right) + \frac{1}{r_1}\right)^2}{1 - \sum \frac{1}{r}}.$$

It is fairly easy to see that this maximum is achieved for $r_1 = 2, r_2 = 3$, and $r_3 = 7$, and is equal to $\frac{23^2}{42} = 12\frac{25}{42}$. Cases (b) and (c) are checked by similar (but easier) arithmetical calculations.
The maximum for the expression (d) for $h^0(D)$ is also achieved in the same case and is equal to 7.

The following example shows that inequalities in Theorem 2.3 are sharp.

Example 2.4. It is not difficult to construct a nonsingular K3 surface $\tilde{S}$ with a set of (-2)-curves that generate $\operatorname{Pic}(\tilde{S}) \otimes Q$ and form the following graph ($\tilde{E}_9$).



After the contraction of curves corresponding to the black vertices, we get a K3 surface S with Du Val singularities and $\rho(S) = 1$; the image of the white vertex is a smooth rational curve C on S, passing through 3 cyclic singular points with invariants (2, 1), (3, 1), (7, 1). Now the linear system $|D| = |23C|$ satisfies all conditions of Theorem 2.2; therefore, $h^0(23C) = h^0(22C) = h^0(21C)$.

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