EXPLICIT COMPACTIFICATIONS OF MODULI SPACES OF CAMPEDELLI AND BURNIAT SURFACES

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ABSTRACT. We describe the compactifications obtained by adding slc surfaces X with ample K_X, for two connected components in the moduli space of surfaces of general type: Campedelli surfaces with $\pi_1(X) = \mathbb{Z}_2^3$, and Burniat surfaces with $K_X^2 = 6$.

This is the color version; black-and-white version at `http://www.math.uga.edu/~valery/ap-bw.pdf`

CONTENTS

Introduction

Twenty years ago Kollár and Shepherd-Barron [KSB88] proposed a way to compactify the moduli space of surfaces of general type by adding surfaces X with slc (semi log canonical) singularities and ample K_X, *stable surfaces,* similar to the stable curves in dimension 1. This construction was later extended to stable pairs (X, B) and stable maps $f : (X, B) \to V$ [Ale96b, Ale96a], see [Ale06] for more details. Since then, many explicit compactifications of this type were constructed for pairs (X, B) in which the variety X is relatively simple: toric, abelian or spherical [Ale02, AB06], a projective space [Hac04, HKT06, Ale08], a del Pezzo surface [HKT07].

However, in the original case of surfaces of general type not a single explicit compactification was computed, except for the trivial examples of rigid surfaces, products, and symmetric squares of curves; only the theoretical existence of such a compactification was known. (There is also an unpublished result of the first

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author saying that the theta divisors of principally polarized stable semiabelic varieties are slc. This provides the compactification for the moduli space of theta divisors of principally polarized abelian varieties.)

One reason for this is that the situation is easiest when K_X+B is very close to zero; the case of hyperplane arrangements is an exception to this rule. For a smooth surface S of general type, however, $K_S^2 \geq 1$. If a stable surface X has irreducible components X_j then $K_X^2 = \sum (K_{X_j}+D_j)^2$, where D_j is the double locus. The positive number $(K_{X_j}+D_j)^2$ is only rational, and although there is an explicit bound from below, it is very small. So a stable degeneration of S may have a huge number of irreducible components.

The purpose of this paper is to describe explicitly the compactifications of two connected components in the moduli space of surfaces of general type: of Campedelli surfaces with $\pi_1(S)=\mathbb{Z}_2^3$, and of Burniat surfaces with $K_S^2=6$. We also consider an infinite series of surfaces and higher-dimensional varieties generalizing Campedelli surfaces; for these, our results are less explicit.

The stable surfaces appearing on the boundary are quite nontrivial, especially in the Burniat case, and provide examples of many interesting features of the general case.

The construction is an application of [Ale08] which provides a stable pair compactification $\overline{\mathrm{M}}_\beta(r,n)$ for the moduli space of weighted hyperplane arrangements $(\mathbb{P}^{r-1}, \sum b_i B_i)$, for any weight $\beta=(b_1,\dots,b_n)$, $0<b_i\leq 1$.

Both Campedelli and Burniat surfaces are Galois $\mathbb{Z}_2^k$ covers $\pi : X \to Y$ of $Y=\mathbb{P}^2$ (resp. of $Y=\mathrm{Bl}_3\,\mathbb{P}^2$) ramified in an arrangement of 7 lines in general position (resp. 9 lines in special position + 3 exceptional divisors). For the canonical class one has $K_X=\pi^*(K_Y+\sum \frac{1}{2}B_i)$. We apply [Ale08] in the case of $\mathbb{P}^2$ and $n=7$ (resp. 9) with the weight $\beta=(\frac{1}{2},\dots,\frac{1}{2})$.

To analyze the surfaces, we use the theory of abelian covers developed by the second author in [Par91]. We also extend it here to the non-normal covers.

Notation 0.1. Throughout this paper, for the moduli problem we work over $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$. Each variety is defined over an algebraically closed field of characteristic $\neq 2$. *lc, slc* and *lt* stand for *log canonical, semi log canonical* and *log terminal*. We recall the definitions in Section 4.

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1. **Campedelli and Burniat surfaces**

Let X be a projective variety with an action by a finite abelian group G and let $\pi : X\to Y$ be the quotient map. In the case when Y is smooth and X is normal, this cover is conveniently encoded by the *building data* $(L_\chi, D_{H,\psi})$ described in [Par91, §2]. Here, the elements χ go over the group of characters $G^*=\operatorname{Hom}(G,\mathbb{G}_m)$, and (H,ψ) go over the pairs H cyclic subgroups of G + generator of H^*.

When $G=\mathbb{Z}_2^k$, $G^*=\mathbb{Z}_2^k$, the only case considered in this paper, the building data become especially easy and consist of the following:

1. for each nonzero $h\in G$, the reduced ramification divisor $D_h\subset Y$, the image of the divisor on X whose points are fixed by h, and
2. for each $\chi\in G^*$, a line bundle L_χ on Y, the eigenspace of χ, so that $X=\operatorname{Spec}_{\mathcal{O}_Y}\oplus_{\chi\in G^*}L_\chi^{-1}$. One has $L_0=\mathcal{O}_Y$.

The building data must satisfy the fundamental relations:

$$\forall \chi,\chi',\quad L_\chi+L_{\chi'} = L_{\chi+\chi'}+\sum_{h\in G}\epsilon_h^{\chi,\chi'}D_h,$$

where $\epsilon_h^{\chi,\chi'} = 1$ if both $\chi(h) = \chi'(h) = -1$ and $\epsilon_h^{\chi,\chi'} = 0$ otherwise. In particular,

$$2L_\chi = \sum_{h\in G,\ \chi(h)=-1} D_h,$$

so that the latter sum of divisors must be divisible by 2. By [Par91, Thm.2.1], if Y is smooth with $h^0(\mathcal{O}_Y) = 1$ then there is a bijection between the normal $\mathbb{Z}_2^k$ Galois covers $\pi : X \to Y$ and the building data (L_χ, D_h) as above.

The surfaces we consider in this paper are particular $\mathbb{Z}_2^k$ Galois covers. The most accurate names for them are: (numerical) Campedelli surfaces with $\pi_1(X) = \mathbb{Z}_2^3$, and Burniat surfaces with $K_X^2 = 6$ (this is the general case, which can be specialized to obtain surfaces with $2 \le K^2 \le 5$, cf. [Bur66]). To save space, in this paper we will call them simply *Campedelli surfaces* and *Burniat surfaces*. They have $p_g = q = 0$ and $\chi(\mathcal{O}_X) = 1$.

Definition 1.1. A Campedelli surface is a $\mathbb{Z}_2^3$ Galois cover of $\mathbb{P}^2$ whose building data is 7 lines D_h ($h \in \mathbb{Z}_2^3\setminus 0$) in general position. Then $L_\chi = \mathcal{O}_{\mathbb{P}^2}(2)$ for $\chi \neq 0$.

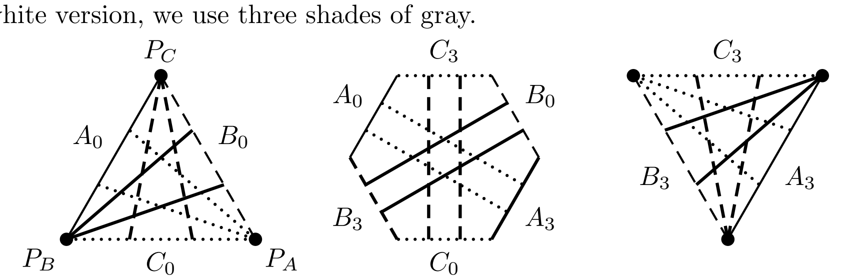
One has $K_X = \pi^*(K_{\mathbb{P}^2}+D)$, $D = \frac{1}{2}\sum D_h$, and so $K_X^2 = 8(\frac{1}{2})^2 = 2$. Campedelli surfaces with fundamental group of order 8 are usually described as free quotients of the intersection of 4 quadrics in $\mathbb{P}^6$ by a group G of order 8 (cf. [MPR09]). When $G = \mathbb{Z}_2^3$, the quadrics can be taken to be diagonal and it is easy to check that the bicanonical system gives a $\mathbb{Z}_2^3$ cover of $\mathbb{P}^2$ branched on 7 lines.

Definition 1.2. By analogy, we define a series of surfaces and higher-dimensional varieties $U(m,k)$ as Galois $\mathbb{Z}_2^k$ covers of $\mathbb{P}^m$ whose building data is $2^k - 1$ hyperplanes D_h ($h \in \mathbb{Z}_2^k \setminus 0$) in general position. One has $L_\chi = \mathcal{O}_{\mathbb{P}^m}(2^{k-2})$ for $\chi \neq 0$. We will call these *Uniform Line (Hyperplane) Cover surfaces (varieties).*

Remark 1.3. Note that the surfaces $U(2,k)$ are smooth (when the lines are in general position). However, such varieties of dimension ≥ 3 are singular over the intersection of three hyperplanes corresponding to three linearly dependent over $\mathbb{F}_2$ elements in G. (Cf. the local analysis in Section 4.2 below.)

The definition of Burniat surface is more involved. First, we consider an arrangement of 9 lines on $\mathbb{P}^2$ shown in the first panel of the following picture. We denote the sides of the triangle A_0, B_0, C_0 and the vertices P_A, P_B, P_C. The point P_A is the point of intersection of B_0 and C_0, etc. There are additional lines A_1, A_2 through P_B, lines B_1, B_2 through P_C, and lines C_1, C_2 through P_A. We assume that the lines are in general position otherwise.

In the color version of this paper the A lines are drawn in red, B lines in blue, and C lines in black. In the black-and-white version, we use three shades of gray.



Now blow up the points P_A, P_B, P_C and denote the resulting exceptional divisors A_3, B_3, C_3. Note that this arrangement can be presented as the blowup of $\mathbb{P}^2$ in a different way by contracting A_0, B_0, C_0. The two line arrangements differ by a Cremona transformation.

Definition 1.4. The Burniat surface is the $\mathbb{Z}_2^2$ cover of $\Sigma = \mathrm{Bl}_3\,\mathbb{P}^2$ for the building data $D_a = \sum_{i=0}^3 A_i$, $D_b = \sum_{i=0}^3 B_i$, $D_c = \sum_{i=0}^3 C_i$, where a, b, c are the 3 nonzero elements of $\mathbb{Z}_2^2$.

When the lines are chosen generically so that on Σ only two divisors intersect at a time (and they belong to different elements of G which is always true for Burniat arrangements), the Galois cover is smooth. Let $D = \frac{1}{2}\sum_{i=0}^3(A_i + B_i + C_i)$. Then $K_\Sigma + D = -\frac{1}{2}K_\Sigma$ is ample, $K_X = \pi^*(K_\Sigma + D)$, and so $K_X^2 = 4\cdot\frac{6}{4} = 6$.

What makes the above surfaces especially interesting is the following fact:

Theorem 1.5. *(over $\mathbb{C}$) The Campedelli and Burniat surfaces form connected components in the moduli spaces of smooth surfaces of general type, of dimensions 6 and 4 respectively.*

The Uniform Line Cover surfaces form an irreducible component of the moduli space of surfaces of general type, of dimension $2(2^k - 5)$.

For Burniat surfaces, this theorem is the title and the main result of [MP01]. For Uniform Line Cover surfaces, the main result of [FP97] or [Par91, Ex.5.1] prove that these surfaces form an irreducible component of the moduli space. In the case of Campedelli, as we have recalled above, every numerical Campedelli surface with fundamental group $\mathbb{Z}_2^3$ can be obtained as a cover of the plane branched on 7 lines, so in this case we actually have a connected component.

Notice that both in the Burniat and the Campedelli case the abelian cover is given by the bicanonical map.

2. Compactifying the moduli of line arrangements

2.1. **Campedelli arrangements.** A Campedelli surface depends on an arrangement of 7 lines $B_1, \dots, B_7$ in general position on $\mathbb{P}^2$. We can consider the moduli of marked arrangements, with labels corresponding to the nonzero elements of $\mathbb{Z}_2^3$. Then the moduli space of unmarked arrangements will be the quotient space of this moduli space by the finite group $\mathrm{GL}(3, \mathbb{F}_2)$. We will stay with the marked moduli since it is easier and there is a universal family over it.

[Ale08] provides a compactification for this moduli space. Namely, it gives a projective scheme $\overline{\mathrm{M}}_{(\frac{1}{2},\dots\frac{1}{2})}(3,7)$ and a family $(\mathcal{Y}, \mathcal{B}_i)$ over it such that every geometric fiber $(Y, \sum \frac{1}{2}B_i)$ is a stable pair.

Theorem 2.1. $\overline{\mathrm{M}}_{(\frac{1}{2},\dots\frac{1}{2})}(3,7)$ *is a* 6*-dimensional normal variety. It coincides with the GIT quotient* $(\mathbb{P}^2)^7 // \mathrm{PGL}(2)$ *for the symmetric choice of the line bundle* $\mathcal{O}(1,\dots,1)$*, which also equals the GIT quotient* $\mathrm{Gr}(3,7)//T$ *of the grassmannian by the torus* $T = \mathbb{G}_m^7 / \operatorname{diag} \mathbb{G}_m$ *for the symmetric choice of linearization. In the family of pairs over* $\overline{\mathrm{M}}_{(\frac{1}{2},\dots\frac{1}{2})}(3,7)$ *every surface* Y *is isomorphic to* $\mathbb{P}^2$.

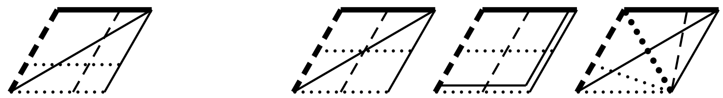
Proof. By [Ale08, Thm.1.5] the moduli space for the weight $(\frac{3}{7}+\epsilon, \dots, \frac{3}{7}+\epsilon)$ coincides with the above GIT quotient. The point $(\frac{1}{2}, \dots, \frac{1}{2})$ lies in the closure of the same chamber as $(\frac{3}{7}+\epsilon, \dots, \frac{3}{7}+\epsilon)$ and it lies strictly above, so by [Ale08, Thm.1.4] the moduli spaces are the same. Finally, for every fiber Y [Ale08, Thm.1.4(3)] says that there is a birational contraction $\mathbb{P}^2 \to Y$, which must be an isomorphism. □

There is another way to see that $\mathbb{P}^2$ never splits into several components in this case. We have $K_{\mathbb{P}^2} + \frac{1}{2}\sum_{i=1}^7 B_i = \frac{1}{2}H$, and it has square $\frac{1}{4}$. On the other hand, by [Ale08, Thm.1.1] the divisor $2(K_Y + \sum \frac{1}{2}B_i)$ must be Cartier and ample, so have an integral positive square. If $Y = \cup Y_j$ has irreducible components Y_j then $(K_Y + B)^2 = \sum (K_Y + B|_{Y_j})^2$. So there can be only one component, and by the general theory of [Ale08] it must be $\mathbb{P}^2$.

Remark 2.2. One may wonder what is the limit of a one-parameter family in which three lines are trying to coincide. The answer is that they don't. Instead, the four complementary lines pass through a common point in the limit. Similarly, five lines do not pass through a common point; instead, the complementary two lines coincide.

For the Uniform Hyperplane arrangements, [Ale08] provides a compactification $\overline{\mathrm{M}}_{(\frac{1}{2},\dots,\frac{1}{2})}(m, 2^k - 1)$ as well. When $m = 2^{k-1} - 2$, the moduli space is a normal variety, a GIT quotient, by the same argument as above. Other than this special case, we don't have much to add to the existence result at the moment, but we plan to study it in the future.

2.2. **Degenerations of Burniat arrangements.** As a warm-up, consider the degenerations of the following pair $(Y, \sum b_i B_i)$ which will be a building block for some of the degenerations below. The surface is $\mathbb{P}^1 \times \mathbb{P}^1$ and there are 7 divisors. Three of them are sections s, three divisors are fibers f, and the 7th divisor has numerical type $s + f$. The two divisors depicted as having double line width are considered with weight 1, the others with weight $\frac{1}{2}$. Note that $K + \sum b_i B_i = \frac{1}{2}s + \frac{1}{2}f$ is ample.



What are the degenerations of this arrangement as stable pairs, i.e. with slc singularities and ample $K_Y + \sum b_i B_i$? The answer is obvious. The arrangement with the first 6 lines is unique, can be identified with $(\mathbb{P}^1, 0, 1, \infty)^2$, and has trivial automorphism group. The diagonal curve has equation $ax_0x_1 - by_0y_1 = 0$, or in affine coordinates $y = cx$, and thus depends on one parameter $c \in \mathbb{A}^1 \setminus 0$. The compactified moduli space is identified with $\mathbb{P}^1$. The arrangements for $c = 1$, 0 and ∞ are shown in the second panel. The last surface is a union of two $\mathbb{P}^2$. The whole situation is toric.

We now consider the degenerations of Burniat arrangements of curves on $\Sigma = \mathrm{Bl}_3\,\mathbb{P}^2$. Some of them are lc, and so "harmless". For example, this is the case when the curve A_1 degenerates into the union $A_0 + C_3$. (Note that the corresponding line arrangement on $\mathbb{P}^2$ in this case is *not* lc, but on Σ there is no problem.)

Then, there are serious degenerations, when the arrangement on Σ is not lc, and so the limit stable surface of a degenerating one-parameter family splits into several irreducible components. Below we list all 10, up to the action of the symmetry group, such degenerations, in addition to the trivial one when Σ does not degenerate. The symmetry group $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_6$ acts by relabeling $A_1 \leftrightarrow A_2$, $B_1 \leftrightarrow B_2$, $C_1 \leftrightarrow C_2$, and rotating the hexagon. (The 180° rotation corresponds to the Cremona transformation.) We list only one degeneration in every orbit.

Notation 2.3. Consider a divisor $D = \sum_{i=0}^{3}(a_iA_i + b_iB_i + c_iC_i)$ on Σ such that $K_\Sigma + D$ is linearly equivalent to zero, and assume that all coefficients $a_i, b_i, c_i \le \frac{1}{2}$. This means:

$$0 \le a_i \le \tfrac{1}{2}, \quad 0 \le b_i \le \tfrac{1}{2}, \quad 0 \le c_i \le \tfrac{1}{2},$$
$$\textstyle\sum_{i=0}^{2}(a_i + b_i + c_i) = 3, \quad \sum_{i=1}^{3}(a_i + b_i + c_i) = 3,$$
$$a_3 = c_0 + c_1 + c_2 + b_0 - 1,\ b_3 = a_0 + a_1 + a_2 + c_0 - 1,\ c_3 = b_0 + b_1 + b_2 + a_0 - 1.$$

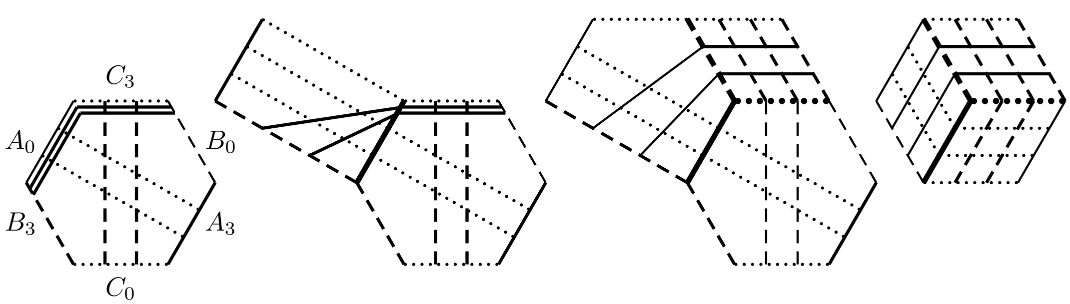
Our label for a degeneration is the set of inequalities that are violated when the pair is not lc.

According to the general theory of [Ale08], these inequalities define a weighted matroid polytope. Tilings by such polytopes then describe the strata in the compactified moduli space.

The cases 1–7 are toric. Every polytope P_j in the answer: the rhombus, trapezoid, triangle, corresponds to a toric variety together with the polarization $2(K_Y + B)|_{Y_j}$: $(\mathbb{F}_0, \mathcal{O}(1,1))$, $(\mathbb{F}_1, \mathcal{O}(s + 2f))$, $(\mathbb{P}^2, \mathcal{O}(1))$. Here, $\mathbb{F}_n$ denotes the standard rational ruled surface with an exceptional section s_n of square $-n$ and a fiber f.

Case 1. $a_0 + a_1 + a_2 \le 1$, $c_3 + a_1 + a_2 \le 1$

Consider a one-parameter family over a smooth curve $(C, 0)$ in which the surface is $\Sigma \times C$ and the divisors degenerate so that in the central fiber both A_1 and A_2 become $A_0 + C_3$, as shown in the picture.



Blow up the line A_0 in the central fiber. Then the central fiber becomes $\Sigma \cup \mathbb{F}_1$. Blowing up the strict preimage of C_3 changes $\mathbb{F}_1$ into $\mathrm{Bl}_2\,\mathbb{P}^2$ and inserts $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. To make such computations, we use the well-known *triple point formula:* Let $Y = \cup Y_j$ be the central fiber in a smooth one-parameter family, and assume that Y is reduced and simple normal crossing. Let C be the intersection $Y_1 \cap Y_2$, suppose it is smooth.

Then

$$(C|_{Y_1})^2 + (C|_{Y_2})^2 + (\text{the number of the triple points of } Y \text{ contained in } C) = 0.$$

For the central fiber, the divisor $K_Y + D$ restricted to an irreducible component Y_j is $K_{Y_j} + D|_{Y_j} +$ (the double locus). The curves A_i, B_i, C_i appear in the last sum with coefficient $\frac{1}{2}$, and the curves in the double locus with coefficient 1.

A simple computation shows that after the last step on the central fiber $K_Y + D$ is big, nef and zero on 3 curves. The 3-fold pair $(\mathcal{Y}, \mathcal{D})$ is log terminal. In characteristic zero the Base Point Free Theorem immediately says that a big positive multiple $N(K_{\mathcal{Y}} + \mathcal{D})$ gives a birational morphism contracting the 3 zero curves. Since the situation is so elementary, this is easy to check in any characteristic, with $N = 2$. The resulting 3-fold is the lc model of the degenerate family, the stable limit proposed in [KSB88].

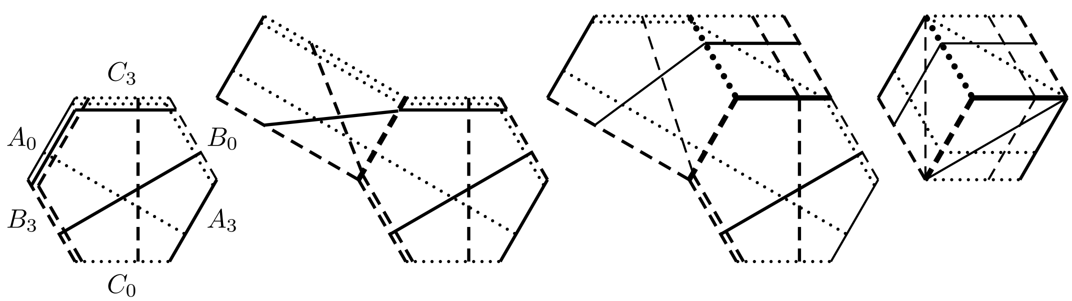
The central fiber is a union of three $\mathbb{P}^1 \times \mathbb{P}^1$ together with 8 lines on each. The moduli space of such generic surfaces is $(M_{0,4})^3 = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^3$. The natural compactification is $(\overline{\mathrm{M}}_{0,(1,\frac{1}{2},\frac{1}{2},\frac{1}{2})})^3 = (\mathbb{P}^1)^3$. Here, $\overline{\mathrm{M}}_{0,(1,\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ is the moduli space of weighted genus 0 curves [Has03]. At the 3 points on the boundary two out of the three curves of weight $\frac{1}{2}$ coincide.

All of these surfaces can be obtained as limits of (Σ, A_i, B_i, C_i). An element of $(\mathbb{P}^1)^3$ is the same as three crossratios given by the three pencils $\Sigma \to \mathbb{P}^1$. The first crossratio is given by the points on the $\mathbb{P}^1$ fibers over which A_1, A_2 are, plus the two singular fibers; similarly for the other two pencils. Now, simply consider the one-parameter family in which B_1, B_2, C_1, C_2 stay fixed, and A_1, A_2 degenerate to $A_0 + C_3$ so that 3 points on $\mathbb{P}^1$ come together while keeping the crossratio fixed.

Remark 2.4. The rules for labeling the double locus are explained in Section 5.1.

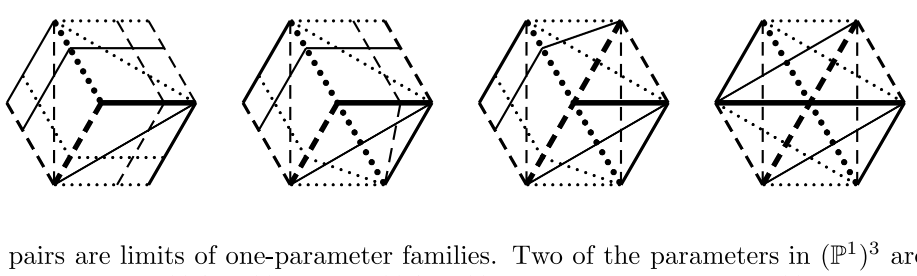
We now run a similar computation in the other 9 cases, without so many words, letting the pictures show the procedure.

Case 2. $a_0 + a_1 + b_2 \leq 1$, $c_3 + c_2 + a_1 \leq 1$



Each of the irreducible components is a $\mathbb{P}^1 \times \mathbb{P}^1$ with 7 lines as in the warm-up at the beginning of this section. Hence, we get a family of pairs over $(\mathbb{P}^1)^3$.

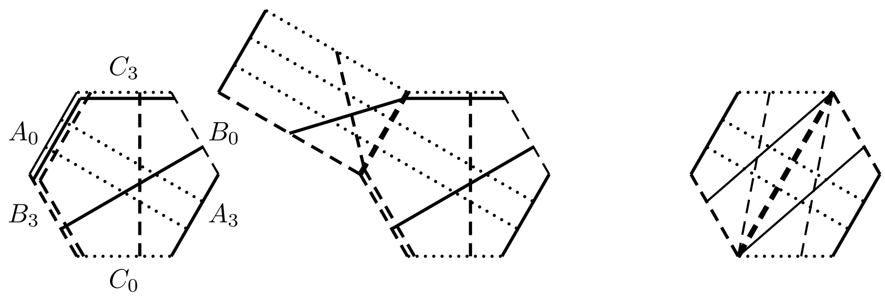
The possible types of degenerate surfaces that appear are the following, where we do not draw all the possibilities for the lines.



All of these pairs are limits of one-parameter families. Two of the parameters in $(\mathbb{P}^1)^3$ are given by the limit values of the ratios $f_{B_2}(t)/f_{A_1}(t)$ and $f_{C_2}(t)/f_{A_1}(t)$ where the functions $f_{B_2}(t)$, etc., measure how fast B_2 approaches A_0, etc. The remaining $\mathbb{P}^1$ is given by the arrangement of the curves A_2, B_1, C_1 that remain on the first irreducible component, the birational image of Σ.

In addition to case 2, we call the new surfaces appearing above cases 3,4,5.

Case 6. $a_0 + a_1 + b_2 \leq 1$

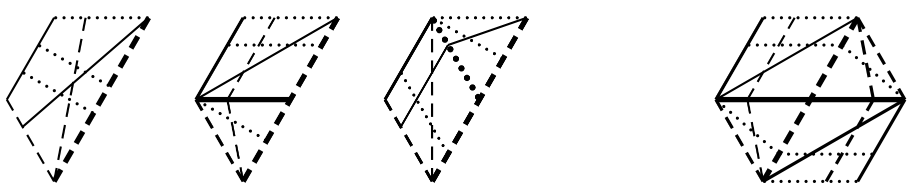


The degenerate surface is $\mathbb{F}_1 \cup \mathbb{F}_1$, the curves A_2, B_1 on the first $\mathbb{F}_1$, and A_1, B_2 on the second $\mathbb{F}_1$, are sections of the numerical type $s_1 + f$.

Fix the two curves C_1, C_2 and vary the curves A_2, B_1 and A_1, B_2. The space of possibilities is $(\mathbb{P}^1)^2$. For the extreme values $0, \infty \in \mathbb{P}^1$ the curve A_2 (resp. B_1) degenerates to $A_3 + C_0$ (resp. $A_3 + B_0$); similarly for the other $\mathbb{P}^1$.

Next, vary the curves C_1 and C_2. This gives 4 points on $\mathbb{P}^1$ along which the irreducible components $\mathbb{F}_1$ intersect. The points of intersection with C_1, C_2 are taken with weight $\frac{1}{2}$ and the other two points with weight 1. The natural compactification of this moduli space of $\mathbb{P}^1$ with 4 weighted points is $\overline{\mathrm{M}}_{0,(1,1,\frac{1}{2},\frac{1}{2})} = \mathbb{P}^1$. For one of the 3 points at the boundary one gets $C_1 = C_2$, and otherwise the above analysis hold.

For the other two points on the boundary of $\overline{\mathrm{M}}_{0,(1,1,\frac{1}{2},\frac{1}{2})}$, when a point of weight 1 collides with a point of weight $\frac{1}{2}$, $\mathbb{P}^1$ splits into two $\mathbb{P}^1$s. A direct computation of the type we performed above shows that for each $\mathbb{F}_1$ there are two possibilities for the limit, shown in the first panel of the following picture.



Each of $\mathbb{P}^1 \times \mathbb{P}^1$ can further degenerate into a union of two $\mathbb{P}^2$, as in the warm-up example. Putting the two halves together, we get several types of reducible surfaces. Most of them appeared in the previous case. The only new one, up to the symmetry, is the surface shown in the last panel of the above picture.

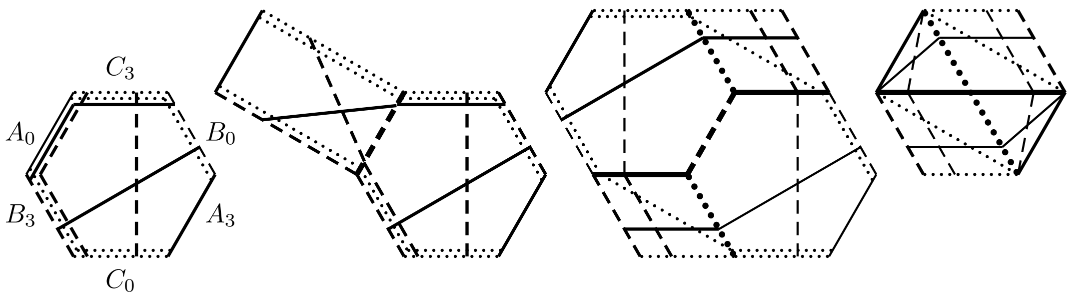
The whole compact stratum of the moduli has a natural morphism to $\overline{\mathrm{M}}_{0,(1,1,\frac{1}{2},\frac{1}{2})} = \mathbb{P}^1$. As we showed above, over any point in $\mathbb{P}^1$ different from $0, \infty$, the fiber is $(\mathbb{P}^1)^2$. The analysis of the degenerations shows that the fibers over 0 and ∞ are isomorphic $(\mathbb{P}^1 \cup_{\mathrm{pt}} \mathbb{P}^1)^2$.

Thus, we obtained a 3-dimension stratum in this way. Again, all of the surfaces parameterized by this stratum are limits of one-parameter families.

We call the new surface that appeared above case 7. Here is how it appears on its own:

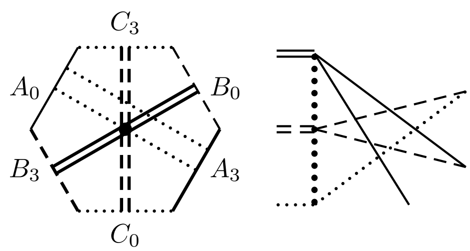
Case 7. $c_3 + c_2 + a_1 \leq 1$, $b_3 + b_2 + c_1 \leq 1$

(Note that these two inequalities imply that $a_0 + a_1 + b_2 \leq 1$.)



Each of the components $\mathbb{P}^1 \times \mathbb{P}^1$ can degenerate into a pair of $\mathbb{P}^2$, as above. As in the warm-up, this gives a family of pairs over $(\mathbb{P}^1)^2$.

Case 8. $a_1 + a_2 + b_1 + b_2 + c_1 \leq 2$



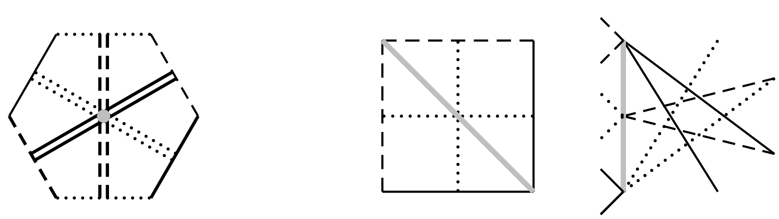
Blowing up the point of intersection of the 5 lines makes the central fiber into a union of $\mathrm{Bl}_1 \Sigma = \mathrm{Bl}_4 \mathbb{P}^2$ and $\mathbb{P}^2$. (We do not draw $\mathrm{Bl}_4 \mathbb{P}^2$.)

For a fixed limit arrangement of lines on Σ, the moduli of the pairs of this type is isomorphic to a codimension 2 closed subset the compactified moduli space of weighted line arrangements $\overline{\mathrm{M}}_{(1,\frac{1}{2},\dots,\frac{1}{2})}(3,6)$ (which is irreducible, by the same argument we used for $\overline{\mathrm{M}}_{(\frac{1}{2},\dots,\frac{1}{2})}(3,7)$ in Section 2.1). So it has dimension 2.

There is a one-parameter family of possible arrangements of curves on Σ of this type. Two of the limit cases are when C_2 degenerates into $C_3 + B_0$, resp. $C_0 + B_3$. This does not add new non lc singularities, and the analysis remains the same. The third limit case is when C_2 degenerates to C_1. This case, which is also a degeneration of case 9, is described in case 10.

Altogether, this gives a 3-dimensional irreducible stratum.

Case 9. $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 \leq 2$



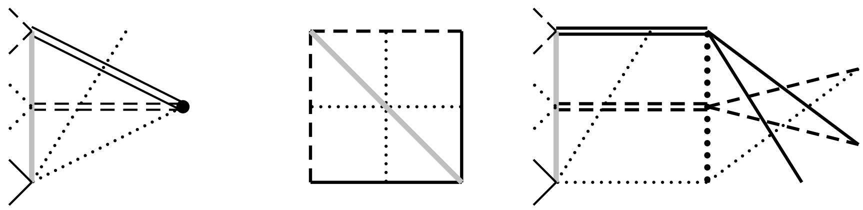
On the blowup of Σ at the intersection point of 3 double lines, the strict preimages of the double lines are (-1)-curves, and they are contracted by $K_{\mathcal{Y}} + \mathcal{D}$ to give a $\mathbb{P}^1 \times \mathbb{P}^1$. The second irreducible component is a $\mathbb{P}^2$. The two components are glued along $\mathbb{P}^1$ which is *not* in the ramification locus.

This stratum isomorphic to a closed subset of codimension 3 of the compactified moduli space of line arrangements $\overline{\mathrm{M}}_{(1,\frac{1}{2},\dots,\frac{1}{2})}(3,7)$. It is easy to see that it has dimension 3.

All of the surfaces parameterized by this stratum are limits of one-parameter families. To see this, for example fix the lines A_1 and B_1. Then the 3-dimensional family is given by the ratios of the speeds of degeneration of the lines A_2, B_2, C_1, C_2 to the point of intersection, similarly to case 2.

Case 10. $a_0 + b_0 + c_0 \leq 1$, $a_1 + a_2 + b_1 + b_2 + c_1 \leq 2$

Finally, the last case is obtained as a common degeneration of the previous two cases: either in case 8 the curve C_2 approaches the point of intersection of the other 5 curves, or in case 9 five lines on the second component $\mathbb{P}^2$ come together to pass through the same point.



The first irreducible component is still $\mathbb{P}^1 \times \mathbb{P}^1$, the second one is $\mathbb{F}_1$, and the third one is $\mathbb{P}^2$.

Together with Σ itself, this gives a total of 11 types of surfaces Y.

2.3. **Compactified moduli of Burniat arrangements.** Here, we consider the moduli $\mathrm{M}_{\mathrm{Bur}}$ of *marked* Burniat arrangements, where the curves are labeled. Over it we have a universal family of pairs (Σ, A_i, B_i, C_i). The moduli of unmarked arrangements where we do not forget the difference between the three groups $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$ is obtained from it by dividing by the group $\mathbb{Z}_2^4$ which acts by relabeling $A_1 \leftrightarrow A_2$, $B_1 \leftrightarrow B_2$, $C_1 \leftrightarrow C_2$, and rotating the hexagon by 180° (corresponding to the Cremona transformation). If we also divide by $\mathrm{Aut}(\mathbb{Z}_2^2) = S_3$, we need to divide by $\mathbb{Z}_2^3 \rtimes (S_3 \oplus \mathbb{Z}_2)$.

Theorem 2.5. *There exist a 4-dimensional irreducible projective scheme* $\overline{M}_{\mathrm{Bur}}$ *and a family* $(\mathcal{Y}, \mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i)$*,* $i = 0, 1, 2, 3$*, over it with the following properties:*

1. *Each geometric fiber* (Y, D)*, where* $D = \frac{1}{2}\sum_{i=1}^{3}(A_i + B_i + C_i)$*, is a stable pair:* (Y, D) *has slc singularities and* $2(K_Y + D)$ *is an ample Cartier divisor.*
2. *Over an open dense subset* $\mathrm{M}_{\mathrm{Bur}}$ *the family coincides with the family of marked Burniat arrangements* (Σ, A_i, B_i, C_i)*.*

One way to prove this theorem is to consider the closed subscheme in the compactified moduli space $\overline{\mathrm{M}}_{(\frac{1}{2},\dots,\frac{1}{2})}(3, 9)$ provided by [Ale08] together with its universal family. Then one can show that

1. The points P_A, P_B, P_C (introduced in the first picture) on the stable limits Y' of the line arrangements remain in the nonsingular part.
2. Denoting, as before, by A_3, B_3, C_3 the exceptional divisors of the blowup $f : Y'' \to Y'$ at P_A, P_B, P_C, the divisor $f^*K_{Y'} - \frac{1}{2}A_3 - \frac{1}{2}B_3 - \frac{1}{2}C_3$ is big, nef, and its log canonical model is a stable pair (Y, D), $D = \frac{1}{2}\sum_{i=1}^{3}(A_i + B_i + C_i)$ that we are after.

The problem with this approach is that one then has to analyze all the degenerations of the line arrangement $(\mathbb{P}^2, \frac{1}{2}\sum_{i=0}^{2}(A_i + B_i + C_i))$. Many of these may be non-lc on $\mathbb{P}^2$ but lc on Σ. So the degeneration of $\mathbb{P}^2$ may be reducible, but the degeneration of Σ still be Σ. The first such example appears when $A_1 = A_0$ in the limit.

There are many more degenerations of line arrangements than degenerations of Burniat arrangements on Σ. Although we have enumerated them all with a computer, presenting the results is quite space-consuming. So for an easier proof, we adapt the methods of [Ale08] from line arrangements directly to the case of $(\Sigma = \mathrm{Bl}_3 \mathbb{P}^2, \frac{1}{2}\sum_{i=1}^{3}(A_i + B_i + C_i))$.

We start as in [Ale08] with the grassmannian $\mathrm{Gr}(3, 9)$ and a universal family $U \subset \mathbb{P}^8 \times \mathrm{Gr}(3, 9)$ whose fiber over a point $[V] \in \mathrm{Gr}(3, 9)$ is the projective plane $\mathbb{P}V$. If $\mathbb{P}V$ does not lie in one of the standard coordinate hyperplanes in $\mathbb{P}^8$ then the intersections give 9 lines on it. We denote the hyperplanes in $\mathbb{P}^8$ by $\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i$, $0 \le i \le 2$, and the lines on $\mathbb{P}V$ by A_i, B_i, C_i.

For every arrangement of n lines in $\mathbb{P}^{r-1}$ there is a corresponding to it closed subscheme of $\mathrm{Gr}(r, n)$ describing the hyperplane arrangements of this type and its degenerations. It is given by setting to zero the Plücker coordinates for those r-tuples of hyperplanes whose intersections are not empty. In our case, we define the *Burniat matroid* to be the one given by the generic Burniat arrangement depicted on page 3. Thus, it is given by setting to zero the Plücker coordinates p_{ijk} where ijk is a subset of one of the sets $C_0C_1C_2B_0$, $A_0A_1A_2C_0$, $B_0B_1B_2A_0$. We denote this closed subscheme V_{Bur} and from now on look at the restricted universal family over V_{Bur}.

Now we modify $\mathbb{P}^8$ by blowing up the intersections $\mathbf{C}_0\cap\mathbf{C}_1\cap\mathbf{C}_2\cap\mathbf{B}_0$, $\mathbf{A}_0\cap\mathbf{A}_1\cap\mathbf{A}_2\cap\mathbf{C}_0$, $\mathbf{B}_0\cap\mathbf{B}_1\cap\mathbf{B}_2\cap\mathbf{A}_0$, and denote the corresponding exceptional divisors $\mathbf{A}_3, \mathbf{B}_3, \mathbf{C}_3$. The blowup of the restricted family gives the universal family $U_{\rm Bur} \to V_{\rm Bur}$ whose generic fiber is $\Sigma V \simeq \mathrm{Bl}_3\,\mathbb{P}V$. On the open dense subset where the fiber is not contained in the 12 divisors one obtains 12 divisors A_i, B_i, C_i, $0 \le i \le 3$, on it.

Definition 2.6. The Burniat polytope $\Delta_{\rm Bur}$ is the polytope in $\mathbb{R}^{12}$ with coordinates a_i, b_i, c_i $(i = 0, 1, 2, 3)$ defined by the following equations and inequalities:

$$0 \le a_i \le \tfrac{1}{2}, \quad 0 \le b_i \le \tfrac{1}{2}, \quad 0 \le c_i \le \tfrac{1}{2},$$
$$\textstyle\sum_{i=0}^2 (a_i + b_i + c_i) = 3, \quad \sum_{i=1}^3 (a_i + b_i + c_i) = 3,$$
$$a_3 = c_0 + c_1 + c_2 + b_0 - 1,\ b_3 = a_0 + a_1 + a_2 + c_0 - 1,\ c_3 = b_0 + b_1 + b_2 + a_0 - 1.$$

This polytope can be embedded as a maximal dimensional polytope into $\Delta_{(\frac{1}{2},\dots\frac{1}{2})}(3,9)$ by using the coordinates (a_i, b_i, c_i) with either $0 \le i \le 2$ or $1 \le i \le 3$.

Definition 2.7. A matroid tiling of $\Delta_{\rm Bur}$ is a tiling by the polytopes which are the intersections with $\Delta_{\rm Bur}$ of matroid polytopes in $\Delta(3, 9)$.

We now set up the GIT problem for the universal family $U_{\rm Bur} \to V_{\rm Bur}$. We have a natural action of the torus $T = \mathbb{G}_m^9/\operatorname{diag}\mathbb{G}_m$ on $U \to \mathrm{Gr}(3, 9)$ and $U_{\rm Bur} \to V_{\rm Bur}$. We choose

(1) The $\mathbb{Q}$-line bundle $L = p_1^*\mathcal{O}_{\mathrm{Bl}_3\,\mathbb{P}^8}(\frac{3}{2}H - \frac{1}{2}A_3 - \frac{1}{2}B_3 - \frac{1}{2}C_3)) \otimes p_2^*\mathcal{O}_{V_{\rm Bur}}(1))$ on $U_{\rm Bur} \subset \mathrm{Bl}_3\,\mathbb{P}^8 \times V_{\rm Bur}$. Here, $\mathcal{O}_{V_{\rm Bur}}(1)$ is the restriction of the Plücker line bundle of $\mathrm{Gr}(3, 9)$ and H is the hyperplane on $\mathbb{P}^8$. Note that this $\mathbb{Q}$-line bundle restricts to $K_{\Sigma V} + \frac{1}{2}\sum_{i=0}^3 (A_i + B_i + C_i) = -\frac{1}{2}K_{\Sigma V}$ on each fiber ΣV.
(2) The symmetric linearization of the T-action on L.

Let $[V] \in \mathrm{Gr}_{\rm Bur}(3, 9)$ be a point. Its matroid polytope P_V is the convex hull of the points $(1, 1, 1, 0, \dots 0)$ corresponding to the nonzero Plücker coordinates (see [Ale08, Def.2.6]).

We consider the fiber $\mathrm{Bl}_3\,\mathbb{P}V$ over $[V]$ and a point $p \in \Sigma V$. We want to know when this point is (semi)stable w.r.t the T-action on $U_{\rm Bur}$.

Lemma 2.8. (1) *If $P_V \cap \Delta_{\rm Bur} = \emptyset$ or $\mathrm{Bl}_3\,\mathbb{P}V$ lies in one of the 12 divisors, then no $p \in \mathrm{Bl}_3\,\mathbb{P}V$ is semistable*
(2) *p is semistable $\implies$ the pair $(\mathrm{Bl}_3\,\mathbb{P}V, \frac{1}{2}\sum_{i=0}^3 (A_i + B_i + C_i))$ is lc at p.*

Proof. This is a literal translation of [Ale08, Thm.6.6], with the same proof. □

Next, by analogy with the weighted grassmannian of [Ale08], we define the *Burniat grassmannian.*

Definition 2.9. The Burniat grassmannian is the Proj of the graded subring of the graded ring $V_{\rm Bur}$ whose weights lie in the cone over $\Delta_{\rm Bur}$.

In other words: start with the homogeneous ring of the grassmannian $\mathrm{Gr}(3, 9)$, generated by the Plučker coordinates p_{ijk}. It is $\mathbb{Z}^n$-graded, and all monomials in p_{ijk} lie in the cone over the hypersimplex $\Delta(3, 9)$.

Now set $p_{ijk} = 0$ for all triples ijk which are subsets of one of the sets $C_0C_1C_2B_0$, $A_0A_1A_2C_0$, $B_0B_1B_2A_0$. Then consider the subring generated by the monomials whose weights lie in a smaller cone over $\Delta_{\rm Bur} \subset \Delta(3, 9)$.

Following [Ale08] and modifying it appropriately, we now define the compactified moduli space of Burniat arrangements on Σ.

Definition 2.10. (cf. [Ale08, Def.7.6]) $\overline{\mathrm{M}}_{\rm Bur}$ is the moduli space of stable toric varieties $Z \to \mathrm{Gr}_{\rm Bur}$.

Such stable toric varieties are described by the matroid tilings of $\Delta_{\rm Bur}$. For every tiling, there is a bijective dimension-preserving correspondence between the matroid polytopes and strata of Z, with the maximal-dimensional polytopes giving the irreducible components of Z.

For every such stable toric variety $Z \to \mathrm{Gr}_{\rm Bur}$, let Z^0 be its interior, obtained by removing the divisors corresponding to the boundary of $\Delta_{\rm Bur}$. Note that $\mathrm{Gr}_{\rm Bur}$ and $V_{\rm Bur}$ have the same interior.

Definition 2.11. Let $Y := f^{-1}(Z^0)//T$ be the GIT quotient of the universal family over Z^0 by the torus. It comes with 12 divisors which are the GIT quotients of the 12 divisors $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$ $(0 \le i \le 3)$ on the universal family.

Theorem 2.12. (1) *If Z is irreducible then Y is isomorphic to $\Sigma = \mathrm{Bl}_3\,\mathbb{P}^2$ and the 12 divisors on Y form a lc Burniat arrangement.*
(2) *For any Z, the variety Y is reduced and projective, the pair $(Y, \frac{1}{2}\sum_{i=1}^{3}(A_i + B_i + C_i))$ is slc, and $K_Y + \frac{1}{2}\sum_{i=1}^{3}(A_i + B_i + C_i)$ is ample.*

Proof. A word-for-word translation of the proof of [Ale08, Thm.7.4]. □

We will call such pairs the *stable Burniat pairs*. To describe $\overline{\mathrm{M}}_{\mathrm{Bur}}$ and the pairs over it, we now must describe the matroid polytopes intersecting the interior of Δ_{Bur} and the tilings of Δ_{Bur} by such polytopes.

Every such maximal-dimensional polytope Q_j corresponds to a degenerate line arrangement $(\mathbb{P}^2, F_j)$. Let $X'_j = \mathrm{Bl}_3\,\mathbb{P}^2$ and $F'_j = f^*F_j - \frac{1}{2}A_3 - \frac{1}{2}B_3 - \frac{1}{2}C_3$. Let (Y_j, D_j) be its log canonical model which we define as follows: If $g : X''_j \to X'_j$ is a log resolution and $g^*(K_{X'_j} + F'_j) = K_{X''_j} + \sum a_i E_i$ then we define (Y_j, B_j) to be the log canonical model of $(X''_j, \sum \min(1, a_i)E_i)$.

By the theory of [Ale08], this log canonical model gives an irreducible component (Y_j, D_j) of the stable pair (Y, D). One has $\frac{6}{4} = (K_Y + D)^2 = \sum (K_{Y_j} + D_j)^2$.

Theorem 2.13. *The maximal-dimensional intersections of matroid polytopes with Δ_{Bur} are given in Table 1. For each of them we list the log canonical model (Y_j, D_j) and $(K_{Y_j} + D_j)^2$. We list only one polytope in each $(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_6)$-orbit. The cases refer to Section 2.2.*

TABLE 1. Maximal-dimensional intersections of matroid polytopes with Δ_{Bur}

No	Inequalities	Case	Y_j	$4(K_{Y_j} + D_j)^2$
0	Δ_{Bur}		Σ	6
1	$a_0 + a_1 + a_2 \le 1$ $c_3 + a_1 + a_2 \le 1$	1	$\mathbb{P}^1 \times \mathbb{P}^1$	2
2	$a_0 + a_1 + b_2 \le 1$ $c_3 + c_2 + a_1 \le 1$	2	$\mathbb{P}^1 \times \mathbb{P}^1$	2
3	$a_0 + a_1 + b_2 \le 1$	6	$\mathbb{F}_1$	3
4	$c_3 + c_2 + a_1 \le 1$ $b_3 + b_2 + c_1 \le 1$	7	$\mathbb{P}^2$	1
5	$a_1 + a_2 + b_1 + b_2 + c_1 \le 2$	8	$\mathrm{Bl}_4\,\mathbb{P}^2$	5
6	$a_0 + b_0 + c_0 + c_2 \le 1$	8	$\mathbb{P}^2$	1
7	$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 \le 2$	9	$\mathbb{P}^1 \times \mathbb{P}^1$	2
8	$a_0 + b_0 + c_0 \le 1$	9	$\mathbb{P}^2$	4
9	$a_0 + b_0 + c_0 \le 1$ $a_1 + a_2 + b_1 + b_2 + c_1 \le 2$	10	$\mathbb{F}_1$	3

Proof. This is a straightforward case-by-case analysis, very much as in Section 2.2. It is helped by the following considerations: The three lines A_0, B_0, C_0 may come together but the three points P_A, P_B, P_C must remain distinct, otherwise $K + D$ is not big. Similarly, 4 divisors on Σ cannot have a 1-dimensional intersection. The rest is quite easy. □

Theorem 2.14. *If (Y, D) is a stable Burniat pair then either $Y = \Sigma$ and (Y, D) is an lc pair or Y is one of the 10 surfaces described in Section 2.2.*

Proof. Again, this is a straightforward case-by-case search helped by the identity $\frac{6}{4} = (K_Y + D)^2 = \sum (K_{Y_j} + D_j)^2$. If two of the above polytopes share a facet then they should have complementary inequalities. So the search is quite easy to do by hand (and then we confirmed it with a computer). □

End of proof of Theorem 2.5. The only part that is not proved yet is that $\overline{M}_{\mathrm{Bur}}$ is irreducible. This follows from the fact that, as we checked in Section 2.2, every surface appearing in the strata is a limit of a nonsingular arrangement. □

3. Non-normal $\mathbb{Z}_2^k$ covers $f : X \to Y$

In this section we extend the theory of abelian covers $\pi : X \to Y$ of [Par91] to the case when X and Y are reduced, S_2 and n.c. (normal crossing in codimension 1).

Setup 3.1. X is a reduced projective variety with a faithful action by a finite group G, and $Y = X/G$. We assume that both X and Y satisfy Serre's condition S_2 and that they are n.c.

Let X be a variety (or more generally an equidimensional S_1 Japanese Noetherian scheme) and let F be a coherent sheaf whose every associated component has dimension equal to $\dim X$.

Recall that there exists a unique S_2*-fication,* or *saturation in codimension 2*, a coherent sheaf defined by

$$F^{\mathrm{sat}}(V) = \varinjlim_{U\subset X,\ \mathrm{codim}(X\setminus U)\ge 2} F(V\cap U)$$

F^{sat} is S_2, and F is S_2 iff $F^{\mathrm{sat}} = F$. In particular, for $F = \mathcal{O}_X$ one obtains the S_2-fication, or saturation $X^{\mathrm{sat}} \to X$ which is dominated by the normalization of X.

Lemma 3.2. *Let Y be an S_2 variety, $Y^0 \subset Y$ an open subset with $\mathrm{codim}(Y \setminus Y^0) \ge 2$, and $f^0 : X^0 \to Y^0$ a finite G-cover with S_2 variety X^0. Then there exists a unique S_2 variety X and a G-cover $f : X \to Y$ whose restriction to Y^0 is f^0.*

Proof. For the existence, we take $\mathcal{O}_X := i_*\mathcal{O}_{X^0}$, where $i : Y^0 \to Y$ is the embedding. This is automatically a $\mathcal{O}_Y$-algebra whose Spec gives X. It is also unique by the S_2 condition. □

Now let $f : X \to Y$ be as the setup. By the above lemma, we can always remove codimension 2 closed subsets, keeping the condition $H^0(\mathcal{O}_{X^0}) = H^0(\mathcal{O}_{Y^0}) = k$. Thus we can assume that both X and Y are n.c. and have smooth normalizations.

If X and Y are both normal then we take Y^0 to be smooth. In this case, the cover $f^0 : X^0 \to Y^0$ is uniquely described by the building data $(L_\chi, D_{H,\psi})$ [Par91, Thm.2.1] on Y^0, which for the case $G = \mathbb{Z}_2^k$ we recalled in Section 1. (Note that in the statement of [Par91, Thm.2.1] Y^0 is assumed to be complete but only the condition $H^0(\mathcal{O}_{Y^0}) = k$ is used.)

To translate this building data from Y^0 to Y, we simply need to take L_χ to be in the group $\mathrm{Cl}(Y)$ of Weil divisors modulo linear equivalence.

We now relax the singularity assumptions first on X and then on Y.

3.1. The case of normal Y.

Suppose that Y is normal but X is only n.c. By Lemma 3.2, we can assume that Y is smooth. We now show that the theory of [Par91] extends to this situation when $G = \mathbb{Z}_2^k$ but not for more general groups G.

For every building data $(L_\chi, D_{H,\psi})$, [Par91, Def.2.2] defines a *standard abelian cover* explicitly, by equations. When the ramification divisor $D = \sum D_{H,\psi}$ is reduced, the cover is normal, and the opposite is also true by [Par91, Thm.2.1].

Now consider an *arbitrary* G-cover $\pi : X \to Y$ with X S_2 and n.c., and with Y smooth. We would like to know when such a cover is standard.

As usual we have a decomposition:

$$\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \sum_{\chi\in G^*\setminus 1} L_\chi^{-1}, \tag{1}$$

where L_χ is a line bundle and G acts on L_χ^{-1} via the character $\chi \in G^*$. For $\chi_1, \chi_2 \in G^*$, we denote by μ_{χ_1,χ_2} the map $L_{\chi_1}^{-1} \otimes L_{\chi_2}^{-1} \to L_{\chi_1\chi_2}^{-1}$ induced by the multiplication of $\pi_*\mathcal{O}_X$. We wish to determine the order of vanishing of μ_{χ_1,χ_2} along each irreducible component Δ of the branch divisor D. This is done in [Par91] under the assumption that X be normal, so here we consider only the case that X is singular above Δ. Fix such a Δ and denote by $H \subset G$ the subgroup of the elements that fix $\pi^{-1}(\Delta)$ pointwise. The cover $X/H \to Y$ is generically unramified, hence generically smooth, over Δ. It follows that there is an element of H that exchanges the two branches of X at a general point of $\pi^{-1}(\Delta)$.

Let $\pi' \colon X' \to Y$ be the normalization of X, let H' be the inertia subgroup of Δ for the cover π' and let $\psi' \in (H')^*$ be the corresponding generator. The group H' is cyclic of order $m \ge 1$. Since the normalization map $X' \to X$ is G-equivariant, we have a short exact sequence:

$$0 \to H' \to H \to \mathbb{Z}_2 \to 0. \tag{2}$$

We consider the H-covers $p\colon X \to Z := X/H$ and $p'\colon X' \to X'/H' = Z$ and we study the algebras $\mathcal{A} := p_*\mathcal{O}_{X,\Delta'}$ and $\mathcal{A}' := p'_*\mathcal{O}_{X',\Delta'}$, where Δ' is an irreducible component of the inverse image of Δ in Z.

We distinguish three cases:

Case 1): $H' = \{0\}$.

In this case $H \xrightarrow{\sim} \mathbb{Z}_2$ and X is given locally by $z^2 = t^2$.

Case 2): H' is cyclic of order $2m \geq 4$.

We let $\psi \in H^*$ be the generator such that $\psi|_{H'} = \psi'$. The algebra $\mathcal{A}'$ is generated by elements z, w such that:

$$z^m = tw, \quad w^2 = a \tag{3}$$

where $a \in \mathcal{O}_{Z,\Delta'}$ is a unit and H acts on z via the character ψ and on w via the character ψ^m. The eigenspace corresponding to ψ^j is generated by $z_j := z^j$ for $0 \leq j < m$, and by $z_j := wz^{j-m}$ for $m \leq j \leq 2m-1$. So the subalgebra $\mathcal{A}$ is generated by elements of the form $t^{a_j} z_j$ for suitable $a_j \geq 0$.

Since H fixes $p^{-1}(\Delta')$ pointwise, $\mathcal{A}$ is contained in the subalgebra $\mathcal{B}$ of $\mathcal{A}'$ generated by

$$1, z^m = tw, z_j, \quad 1 \leq j \leq 2m-1, \quad j \neq m.$$

$\mathcal{B}$ can be also generated by z_1, z_{m+1}, with the only relation $az_1^2 = z_{m+1}^2$, hence $\operatorname{Spec}\mathcal{B}$ is n.c. and the map $\operatorname{Spec}\mathcal{B} \to \operatorname{Spec}\mathcal{A}$ is an isomorphism. So $\mathcal{A} = \mathcal{B}$.

Case 3): H' is not cyclic.

In this case m is even and $H = H' \times \mathbb{Z}_2$. We denote by ψ the character $\psi' \times 1$ and by ϕ the character such that $H' = \ker\phi$. $\mathcal{A}'$ is generated by z, w such that:

$$z^m = t, w^2 = a, \tag{4}$$

where $a \in \mathcal{O}_{Z,\Delta'}$ is a unit and H acts on z via the character ψ and on w via the character ϕ. Arguing as in the previous case, one checks that $\mathcal{A}$ is generated by:

$$1, z_1 := z, \dots, z^{m-1}, tw, z_{m+1} := zw, \dots z^{m-1}w.$$

$\mathcal{A}$ can also be generated by z_1, z_{m+1} with the only relation $az_1^2 = z_{m+1}^2$.

Denote by $\varepsilon_{\chi_1,\chi_2}$ the order of vanishing on Δ of the multiplication map μ_{χ_1,χ_2}. Using the above analysis and arguing as in the proof of [Par91, Thm. 2.1], one obtains the following rules (where we use the notations of [Par91, Thm. 2.1]):

Case 1): $\varepsilon_{\chi_1,\chi_2} = 1$ if $\chi_1, \chi_2 \notin H^\perp$,
$\varepsilon_{\chi_1,\chi_2} = 0$ otherwise.

Case 2): For $i = 1, 2$, write $\chi_i|_H = \psi^{\alpha_i}$, $0 \leq \alpha_i < 2m$. Then one has:
$\varepsilon_{\chi_1,\chi_2} = 0$ if $\alpha_1 + \alpha_2 \leq m$,
$\varepsilon_{\chi_1,\chi_2} = 1$ if $m < \alpha_1 + \alpha_2 < 2m$,
$\varepsilon_{\chi_1,\chi_2} = 2$ if $2m \leq \alpha_1 + \alpha_2 \leq 3m$,
$\varepsilon_{\chi_1,\chi_2} = 3$ if $3m < \alpha_1 + \alpha_2 < 4m$.

Case 3): For $i = 1, 2$, write $\chi_i|_H = \psi^{\alpha_i}\phi^{\beta_i}$, where either $\beta_i = 0$ and $0 \leq \alpha_i < m$ or $\beta_i = 1$ and $0 < \alpha_i \leq m$. Then one has:
$\varepsilon_{\chi_1,\chi_2} = 0$ if $\alpha_1 + \alpha_2 < m$, $\beta_1 = \beta_2$, or $\beta_1 \neq \beta_2$ and $\alpha_1 + \alpha_2 \leq m$
$\varepsilon_{\chi_1,\chi_2} = 1$ if $\beta_1 \neq \beta_2$, $\alpha_1 + \alpha_2 > m$,
$\varepsilon_{\chi_1,\chi_2} = 2$ if $\beta_1 = \beta_2 = 1$, $\alpha_1 = \alpha_2 = m$.

Theorem 3.3. *Let Y be a normal projective variety and let $G = \mathbb{Z}_2^k$. Then there is a bijection between*

1. *G-covers $f : X \to Y$ such that X is S_2 and n.c. in codimension 1, and*
2. *standard G-covers for the building data (L_χ, D_h) such that the divisor $\sum D_h$ has multiplicity at most 2 along each irreducible component.*

Proof. In the above analysis the group $\mathbb{Z}_2^k$ appears in case 1) and case 3) for $m = 2$. In both situations the cover is standard.

In case 1), the cover π is standard: Δ appears in D_H with multiplicity 2. In case 3), π is standard for $m = 2$: Δ appears with multiplicity 1 in D_{H_1} and D_{H_2}, where H_1, H_2 are the two nontrivial subgroups of H distinct from H'. So [Par91, Thm.2.1] holds for $\mathbb{Z}_2^k$-covers in this more general setting.

Vice versa, assume that Δ appears in $\sum D_h$ with multiplicity ≤ 2. If mult $= 1$ then the cover is normal over Δ. If mult $= 2$ and Δ appears in one D_h then the cover is given over the generic point of Δ by the equation $z^2 = ut^2$, with u a unit; so is n.c. If mult $= 2$ and Δ appears in D_{h_1} and D_{h_2} then the cover is by the equation $z_1^2 = t$, $z_2^2 = ut$, which is equivalent to $z_2^2 = uz_1^2$, so n.c. □

Remark 3.4. One can check that in the other cases the cover is not standard. Therefore, the correspondence does not hold for the groups other than $\mathbb{Z}_2^k$.

3.2. **The general case.** Let C be the divisorial part of the singular locus of Y, $\nu : Y^\nu \to Y$ be the normalization of Y, and C^ν be the preimage of C on Y^ν. Since Y is n.c., we have an involution ι on C^ν such that $C^\nu/\iota = C$. (If Y is a union of smooth components then C^ν is a union of several pairs of varieties, exchanged by the involution i. In general, some components of C^ν map to themselves.)

Theorem 3.5. *Assume that we are given a G-cover $X^\nu \to Y^\nu$ and let $D_X^\nu \to C^\nu$ be the induced cover. Then X^ν can be glued to an S_2 cover $X \to Y$ if and only if:*

(1) *For each component C_i of C, the inertia subgroup of C_i is the same as the inertia subgroup of $\iota(C_i)$.*
(2) *There exists an involution $j : \widetilde{D}_X^\nu \to \widetilde{D}_X^\nu$ of the normalization of D_X^ν which covers the involution $i : C^\nu \to C^\nu$.*

Proof. The two conditions are clearly necessary. Vice versa, given j we can glue X^ν outside a subset of codimension ≥ 2 to get a G-cover $X^0 \to Y^0$ with $\operatorname{codim}(Y \setminus Y^0) \geq 2$. Then by Lemma 3.2 there exists a unique S_2 extension $X \to Y$. □

Warning 3.6. It may happen that there is no covering involution of D_X^ν but only of its normalization $\widetilde{D}_X^\nu$. Then the double locus of X is obtained from $\widetilde{D}_X^\nu/j$ by some additional gluing in codimension 1 (codimension 2 for X). As a consequence, the preimages in X of the irreducible components of Y are not S_2. But the variety X is S_2. Section 5.4 contains multiple examples of this phenomenon.

The only case of this theorem that we will need is the following: Y is a n.c. union of smooth surfaces, and C is a union of smooth curves C_i. Then the above condition is that for each C_i the two data for the normalization of the cover must be the same.

4. Slc $\mathbb{Z}_2^k$ covers $f : X \to Y$

In this section we classify the slc $\mathbb{Z}_2^k$ covers $\pi : X \to Y$ of surfaces under the assumption that the normalization of Y is smooth, and the double locus plus the ramification divisor on each component of Y is a union of "lines": the components are smooth, and distinct components have distinct tangent directions.

4.1. **Generalities on slc covers.** We first recall the standard definitions for the lc and slc singularities.

Let X be a projective variety, B_j, $i = 1, \dots, n$, be effective Weil divisors on X, possibly reducible, and b_j be some rational numbers with $0 < b_j \leq 1$.

Definition 4.1. Assume that X is a *normal* variety. Then X has a canonical Weil divisor K_X defined up to linear equivalence. The pair (X, B) is called *log canonical* if

(1) $K_X + B$ is $\mathbb{Q}$-Cartier, i.e. some positive multiple is a Cartier divisor, and
(2) for every proper birational morphism $\pi : X' \to X$ with normal X', in the natural formula

$$K_{X'} + \pi_*^{-1}B = \pi^*(K_X + B) + \sum a_i E_i$$

one has $a_i \geq -1$. Here, E_i are the irreducible exceptional divisors of π, and the pullback π^* is defined by extending $\mathbb{Q}$-linearly the pullback on Cartier divisors. $\pi_*^{-1}B$ is the strict preimage of B.

If $\operatorname{char} k = 0$ then X has a resolution of singularities $\pi : X' \to X$ such that $\operatorname{Supp}(\pi_*^{-1}B) \cup E_i$ is a normal crossing divisor; then it is sufficient to check the condition $a_i \geq -1$ for this morphism π only.

Definition 4.2. A pair (X, B) is called *semi log canonical* if

(1) X satisfies Serre's condition S_2,

(2) X has at worst n.c. (normal crossing singularities in codimension one), and no divisor B_j contains any component of this double locus,
(3) some multiple of the Weil $\mathbb{Q}$-divisor $K_X + B$, well defined thanks to the previous condition, is $\mathbb{Q}$-Cartier, and
(4) denoting by $\nu : X^\nu \to X$ the normalization, the pair $(X^\nu,\ (\text{double locus}) + \nu_*^{-1}B)$ is log canonical.

Lemma 4.3. *Let $f : X \to Y$ be a finite morphism between equidimensional varieties, both of which are S_2 and n.c. Let B^X and B^Y be effective $\mathbb{Q}$-divisors, as above, not containing any component of the double loci. Let X^0, Y^0 be open subsets with the complements of* codim ≥ 2*, which are n.c. Suppose that the restriction of f is a finite morphism $f^0 : X^0 \to Y^0$ of degree d.*

Assume that $(f^0)^*(K_{Y^0} + B^{Y^0}) = (K_{X^0} + B^{X^0})$ and that char k *does not divide d. Then*
(1) *$K_Y + B^Y$ is $\mathbb{Q}$-Cartier iff so is $K_X + B^X$.*
(2) *The pair (Y, B^Y) is slc iff so is the pair (X, B^X).*

Proof. (1) If the sheaf $L = \mathcal{O}_Y(N(K_Y + B^Y))$ is invertible then we have a homomorphism $\mathcal{O}_Y(N(K_X + B^X)) = i_*(\mathcal{O}_{X^0}(N(K_{X^0} + B^{X^0}))) \to f^*L$ which is an isomorphism outside of codimension 2. So it must be an isomorphism by the S_2 condition. Similarly, if the sheaf $L' = \mathcal{O}_X(N(K_X + B^X))$ is invertible then the sheaf $= \mathcal{O}_Y(Nd(K_Y + B^Y))$ is isomorphic to the norm of L'.

For (2), we can go to the normalizations $X^\nu \to Y^\nu$. We have

$$K_{X^\nu} + B^{X^\nu} := \nu^*(K_X + B^X) = K_{X^\nu} + \nu_*^{-1}B^X + (\text{double locus}),$$

and similarly for Y. One easily checks by the Hurwitz formula that we still have $(f^\nu)^*(K_{Y^\nu} + B^{Y^\nu}) = K_{X^\nu} + B^{X^\nu}$.

This reduces the proof to the normal case and lc instead of slc. In this case the statement is very well known. By the Hurwitz formula again the log discrepancies $a_i^{\log} := 1 + a_i$ for (X^ν, B^{X^ν}) are proportional to those of (Y^ν, B^{Y^ν}) with the coefficient of proportionality equal to the index of ramification for the corresponding exceptional divisors. So one of them is nonnegative iff the other one is. □

We now restrict ourselves to the most interesting case for us.

Lemma 4.4. *Let X and Y be two S_2 varieties which are n.c. Let $f : X \to Y$ be a $\mathbb{Z}_2^k$ cover and $D = \frac{1}{2}\sum D_h$ be the divisor whose restriction to the normalization Y^ν is the union of the ramification divisors without the double locus. Then*
(1) *K_X is $\mathbb{Q}$-Cartier iff so is $K_Y + D$.*
(2) *X is slc iff so is the pair (Y, D).*

Proof. We recall that by Theorem 3.3, the cover $f^\nu : X^\nu \to Y^\nu$ is standard. The formula

$$(f^\nu)^*(K_{Y^\nu} + D + (\text{double locus})) = K_{X^\nu} + (\text{double locus})$$

is easy to check in this case. We finish by applying the previous lemma. □

4.2. **Slc $\mathbb{Z}_2^k$ covers with smooth surface Y.** We fix the group $G = \mathbb{Z}_2^k$. Here we consider a standard G-cover $\pi\colon X \to Y$ of a smooth surface with total branch divisor D such that the pair (Y, D) $(D = \frac{1}{2}\sum D_h)$ is slc, i.e. the surface X is slc by the above section. We describe in detail the singularities of X under the following additional assumptions:
1) the irreducible components of D are smooth;
2) if two irreducible components of D intersect at a point $y \in Y$, then they have distinct tangents there.

All the possible cases are listed in the three tables below. Each line of the tables describes the singularity of X over a point $y \in Y$ that belongs to k components, $D_1, \dots D_k$ of D, possibly not distinct. The assumption that (Y, D) is slc is equivalent to $k \leq 4$ and no three of the D_i coincide. Whenever the D_i are not all distinct, we assume $D_1 = D_2$. The first digit in the label given to each case is equal to the number k of components through y, followed by a $'$ if $D_1 = D_2$ and by a $''$ if $D_1 = D_2$ and $D_3 = D_4$ (obviously this case occurs only for $k = 4$). So, for instance, $3'.m$ means that y belongs to three components of D, two of which coincide.

The entries in the columns have the following meaning:

- $|H|$: for $i = 1, \dots k$ we let $g_i \in G$ be the element such that D_i is a component of D_{g_i} and we denote by H the subgroup generated by $g_1, \dots g_k$;

- *Relations:* describes the relations between $g_1, \dots g_k$. For instance, 123 means $g_1+g_2+g_3=0$.
- *Singularity:* the notations are mostly standard. $\frac{1}{4}(1,1)$ denotes a cyclic singularity $\mathbb{A}^2/\mathbb{Z}^4$ with weights 1,1. $T_{2,2,2,2}$ denotes an arrangement consisting of four disjoint -2-curves $G_1, \dots G_4$ and of a smooth rational curve F intersecting each of the G_i transversely at one point. The self intersection F^2 is given in the table.

  In the non-normal case (Tables 2 and 3) we use the notations of [KSB88], where Kollár and Shepherd-Barron classified all slc surface singularities over $\mathbb{C}$. We work in any characteristic $\neq 2$ but only the singularities from the list in [KSB88] appear.

  "deg.cusp(k)" means a degenerate cusp (cf. [KSB88, def. 4.20]) such that the exceptional divisor in its minimal semiresolution has k components.
- X^ν: denotes the normalization of X (the entries refer to the cases in Table 1);
- $D_X^\nu \to D_X \to C$: D_X^ν is the inverse image in X^ν of the double curve D_X of X and C is the image of D_X in Y. The symbol Δ denotes the germ of a smooth curve, and Γ_k is the seminormal curve obtained by gluing k copies of Δ at one point. The notation $\Gamma_k \xrightarrow{(a_1,\dots a_k:1)} C$ means that the map restricts to a degree a_i map on the i-th component of Γ_k (we do not specify the a_i when they are all equal to 1);
- $\widetilde{X}$: is the minimal semiresolution of X. We write "n.c." when $\widetilde{X}$ has only normal crossings and "pinch" if it has also pinch points;
- B,C,U: we write B (respectively C, U) if the singularity occurs in a degeneration of a Burniat (respectively Campedelli, Uniform Line Cover) surface. For U, we only consider the case of normal Y.

Theorem 4.5. *The slc covers of* $(Y, D=\frac{1}{2}\sum D_h)$ *with smooth* Y *are listed in Tables 2, 3, 4.*

Table 2. One, two, three, and four reduced components

No.	$\lvert H\rvert$	Relations	Singularity	B,C,U
0.1	1	none	smooth	B,C
1.1	2	none	smooth	B,C
2.1	4	none	smooth	B,C
2.2	2	12	A_1	
3.1	8	none	A_1	C
3.2	4	123	$\frac{1}{4}(1,1)$	B,C
3.3	4	12	A_3	
3.4	2	12,13	D_4	
4.1	16	none	elliptic, $F^2=-4$	U
4.2	8	1234	elliptic, $F^2=-8$	C
4.3	8	123	$T_{2,2,2,2}$, $F^2=-4$	C
4.4	8	12	elliptic, $F^2=-2$	
4.5	4	12,134	$T_{2,2,2,2}$, $F^2=-3$	
4.6	4	12,34	elliptic, $F^2=-4$	
4.7	4	12,13	elliptic, $F^2=-1$	
4.8	2	12,13,14	elliptic, $F^2=-2$	

Since all these singularities can be studied in a similar way, we are just going to explain the method and work out two cases as an illustration. We start with some general remarks:

1) we always assume $G=H$. Indeed, the cover π factors as $X \xrightarrow{\pi_2} X/H \xrightarrow{\pi_1} Y$. The map π_1 is étale near y, while for every $z \in \pi_1^{-1}(y)$ the fiber $\pi_2^{-1}(z)$ consists ony of one point. Since G acts transitively on each fiber of π, it is enough to describe the singularity of X above any point $z \in \pi_1^{-1}(x)$.

2) the cover X is normal over the point y iff D is reduced at y. It is nonsingular at y iff either $k=1$ or $k=2$, $D_1 \neq D_2$, $g_1 \neq g_2$ (cf. [Par91, §3]);

3) the cover X is said to be *simple* if the set $\{g_1, \dots g_k\}$ is a basis of $\lvert H\rvert$ (for instance, X is simple if the g_i are all equal). In this case it is very easy to write down the equations of X (see Case $4', 1$ below): X is a complete intersection, and in particular it is Gorenstein.

Table 3. Double component + zero, one, or two reduced components

No.	$\lvert H\rvert$	Relations	Singularity	X^ν	$D_X^\nu \to D_X \to C$	$\widetilde{X}$	B,C
$2'.1$	4	none	semismooth	2(1.1)	$2\Delta \to \Delta \to \Delta$	n.c.	B,C
$2'.2$	2	12	semismooth	2(0.1)	$2\Delta \to \Delta \to \Delta$	n.c.	B
$3'.1$	8	none	semismooth	2(2.1)	$2\Delta \to \Delta \xrightarrow{2:1} \Delta$	n.c.	C
$3'.2$	4	123	[KSB88, 4.23(iii)]	2(2.2)	$2\Delta \to \Delta \to \Delta$	n.c.	B,C
$3'.3$	4	12	semismooth	2(1.1)	$2\Delta \to \Delta \xrightarrow{2:1} \Delta$	n.c.	B
$3'.4$	4	13	semismooth	(2.1)	$\Delta \xrightarrow{2:1} \Delta \to \Delta$	pinch	
$3'.5$	2	12,13	semismooth	(1.1)	$\Delta \xrightarrow{2:1} \Delta \to \Delta$	pinch	
$4'.1$	16	none	deg.cusp(2)	2(3.1)	$2\Gamma_2 \to \Gamma_2 \xrightarrow{(2.2:1)} \Delta$	n.c.	U
$4'.2$	8	1234	deg.cusp(2)	2(3.2)	$2\Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	C
$4'.3$	8	123	$(4'.1)/\mathbb{Z}_2$	2(3.3)	$2\Delta \to \Delta \xrightarrow{2:1} \Delta$	n.c.	C
$4'.4$	8	134	$(4'.1)/\mathbb{Z}_2$	(3.1)	$\Gamma_2 \xrightarrow{2:1} \Gamma_2 \to \Delta$	pinch	C
$4'.5$	8	13	deg.cusp(1)	(3.1)	$\Gamma_2 \to \Delta \xrightarrow{2:1} \Delta$	n.c.	
$4'.6$	8	12	deg.cusp(2)	2(2.1)	$2\Gamma_2 \to \Gamma_2 \xrightarrow{(2,2:1)} \Delta$	n.c.	
$4'.7$	8	34	deg.cusp(6)	2(3.3)	$2\Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	
$4'.8$	4	13,124	$(4'.5)/\mathbb{Z}_2$	(3.3)	$\Delta \xrightarrow{2:1} \Delta \to \Delta$	pinch	
$4'.9$	4	12,134	$(4'.6)/\mathbb{Z}_2$	(2.1)	$\Gamma_2 \to \Delta \xrightarrow{2:1} \Delta$	pinch	B
$4'.10$	4	13,24	deg.cusp(1)	(3.2)	$\Gamma_2 \to \Delta \to \Delta$	n.c.	
$4'.11$	4	12,34	deg.cusp(2)	2(2.2)	$2\Gamma_2 \to \Delta \xrightarrow{(2:1} \Delta$	n.c.	
$4'.12$	4	12,13	deg.cusp(1)	(2.1)	$\Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	
$4'.13$	4	13,14	deg.cusp(3)	(3.3)	$\Gamma_2 \to \Delta \to \Delta$	n.c..	
$4'.14$	2	12,13,14	deg.cusp(1)	(2.2)	$\Gamma_2 \to \Delta \to \Delta$	n.c.	

Table 4. Two double components

No.	$\lvert H\rvert$	Relations	Singularity	X^ν	$D_X^\nu \to D_X \to C$	$\widetilde{X}$	B,C
$4''.1$	16	none	deg.cusp(4)	4(2.1)	$4\Gamma_2 \to \Gamma_4 \xrightarrow{(2,2,2,2:1)} \Gamma_2$	n.c.	U
$4''.2$	8	1234	deg.cusp(4)	4(2.2)	$4\Gamma_2 \to \Gamma_4 \to \Gamma_2$	n.c.	C
$4''.3$	8	123	$(4''.1)/\mathbb{Z}_2$	2(2.1)	$2\Gamma_2 \to \Gamma_3 \xrightarrow{(2,1,1:1)} \Gamma_2$	pinch	C
$4''.4$	8	13	deg.cusp(2)	2(2.1)	$2\Gamma_2 \to \Gamma_2 \xrightarrow{(2,2:1)} \Gamma_2$	n.c.	
$4''.5$	8	12	deg.cusp(4)	4(1.1)	$4\Gamma_2 \to \Gamma_4 \xrightarrow{(2,2,1,1:1)} \Gamma_2$	n.c.	
$4''.6$	4	13,124	$(4''.4)/\mathbb{Z}_2$	(2.1)	$\Gamma_2 \xrightarrow{(2,2:1)} \Gamma_2 \to \Gamma_2$	pinch	
$4''.7$	4	12,134	$(4''.5)/\mathbb{Z}_2$	2(1.1)	$2\Gamma_2 \xrightarrow{(2,2,1,1:1)} \Gamma_3 \xrightarrow{(1,1,2:1)} \Gamma_2$	pinch	B
$4''.8$	4	13,24	deg.cusp(2)	2(2.2)	$2\Gamma_2 \to \Gamma_2 \to \Gamma_2$	n.c.	
$4''.9$	4	12,34	deg.cusp(4)	4(0.1)	$4\Gamma_2 \to \Gamma_4 \to \Gamma_2$	n.c.	B
$4''.10$	4	12,13	deg.cusp(2)	2(1.1)	$2\Gamma_2 \to \Gamma_2 \xrightarrow{(2,1:1)} \Gamma_2$	n.c.	B
$4''.11$	2	12,13,14	deg.cusp(2)	2(0.1)	$2\Gamma_2 \to \Gamma_2 \to \Gamma_2$	n.c.	

The first line in each section of the tables corresponds to the case in which $g_1, \dots g_k$ are a basis of H, hence to a simple cover. All the remaining cases are quotients of an X of this type by a group H_0 that can be read off the "Relations" column. Using the local equations defining X, one can write down a local generator σ of Ω_X and check that σ is invariant under H_0 iff all the relations have even length. (See the analysis of Case $4'.3$ below for an example).

4) the double curve D_X maps onto the divisors that appear in D with multiplicity > 1. Since for a semismooth surface the double curve is locally irreducible, X is never semismooth in the cases $4''$. In

addition, if X is semismooth then the pull back D^ν_X of D_X to the normalization is smooth. Using this remark, it is easy to check that X is never semismooth in the cases $4'$, either.

5) in order to compute the minimal semiresolution $\tilde{X}$, we consider the blow up $Y' \to Y$ of Y at y, pull back X and normalize along the exceptional curve E to get a cover $X' \to Y'$. The branch locus of X' is supported on a s.n.c. divisor and, by construction, the singularities of X'' are only of type 1, 2 or $3'$. Looking at the tables, one sees that either X' is semismooth or it has points of type 2.2 or $3'.2$. In the former case X' is the minimal semiresolution. In the latter case, blowing up Y' at the non semismooth points and taking base change and normalization along the exceptional divisor, one gets a semismooth cover $X'' \to Y''$. The semiresolution $X'' \to X$ is minimal, except in cases $4''.5$, $4''.10$. In these cases the minimal semiresolution $\widetilde{X}$ is obtained by contracting the inverse image in X'' of the exceptional curve of the blow up $Y' \to Y$.

Next we analyze in detail two cases:

Case $4'.1$: By the normalization algorithm ([Par91, §3]), the normalization X^ν is the H-cover branched on $D_1 = D_{g_1+g_2}$, $D_3 = D_{g_3}$ and $D_4 = D_{g_4}$. So g_1 acts on X without fixed points and X is the disjoint union of two copies of the cover 3.1. We choose local analytic coordinates u, v on y such that $D_1 = D_2$ is given locally by $u = 0$, D_3 is defined by $v = 0$ and D_4 by $u + v = 0$. The cover X is defined locally above y by the following equations:

$$z_1^2 = u, z_2^2 = u, z_3^2 = v, z_4^2 = u + v. \tag{5}$$

In particular X is a complete intersection (see remark 3) above). The element g_i acts on z_j as multiplication by $(-1)^{\delta_{ij}}$. The double curve D_X is the inverse image of $u = 0$, hence it is defined by $z_1 = z_2 = 0, z_3 = \pm z_4$ and the map $D_X \to D_1$ is given by $z_3 \mapsto z_3^2$, so D_X is isomorphic to Γ_2, with each component mapping 2-to-1 to $D_1 \simeq \Delta$. The curve D^ν_X is the inverse image of D_1 in X^ν, so it has two connected components, each isomorphic to Γ_2, that are glued together in the map $X^\nu \to X$.

To compute the minimal semiresolution, consider the blow up $Y' \to Y$ of Y at y and the cover $X' \to Y'$ defined in 5). The building data for X' are $D_{g_1+g_2+g_3+g_4} = E$, where E is the exceptional curve of $Y' \to Y$, and, for $i = 1, \dots 4$, $D_{g_i} = D'_i$, where $'$ indicates the strict transform. The cover is singular precisely above $D'_1 = D'_2$ and it is easy, using the local equations, to check that it is n.c. there. So X' is the minimal semiresolution of X. The exceptional divisor is the inverse image F of E in X. Applying the normalization algorithm to the restricted cover $F \to E$, one sees that the normalization F^ν of F is the union of two smooth rational curves F_1 and F_2. The map $F^\nu \to F$ identifies the two points of F_1 that lie over the point $E \cap D'_1$ with the corresponding two points of F_2. Hence X' is the minimal semiresolution of X and the singularity is a degenerate cusp solved by a cycle of two rational curves.

Case $4'.3$: As in the previous case, X^ν and D^ν_X can be computed by the normalization algorithm. One obtains that X^ν is the disjoint union of two copies of 3.3 and D^ν_X is the disjoint union of two copies of Δ. This singularity is the quotient of a cover X_0 of type $4'.1$ by the element $g := g_1 + g_2 + g_3$ Eliminating u and v in (5), we get the following local equations for X_0:

$$z_1^2 - z_2^2 = 0, \quad z_4^2 - z_1^2 - z_3^2 = 0.$$

In these coordinates g acts by $(z_1, z_2, z_3, z_4) \mapsto (-z_1, -z_2, -z_3, z_4)$. A local generator of ω_{X_0} is the residue on X of the differential form

$$\frac{dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4}{(z_1^2 - z_2^2)(z_4^2 - z_1^2 - z_3^2)},$$

which is multiplied by -1 under the action of g. This shows that X is not Gorenstein.

Since the only fixed point of g on X is $x := \pi^{-1}(y)$, the double curve D_X is the quotient of the double curve D_{X_0} of X_0. The two components of D_{X_0} are identified by g, thus D_X is irreducible and maps 2-to-1 onto D_1.

To compute the minimal semiresolution, again we blow up $Y' \to Y$ at y and consider the cover $X' \to Y'$ obtained by pull back and normalization along the exceptional curve E. As usual, we denote by C' the strict transform on Y' of a curve C of Y. The building data for X' are $D_{g_1} := D'_1$, $D_{g_2} = D'_2$, $D_{g_1+g_2} = D'_3$, $D_{g_4} = D'_4 + E$. So X' has normal crossings over D'_1, it has four A_1 points over the point $y' := D'_4 \cap E$ and it is smooth elsewhere (cf. the tables). We blow up at y' and take again pull back and normalization along the exceptional curve E_2. We obtain a cover $X'' \to Y''$ which is nc over the strict transform D''_1 of D'_1 and has no other singularity, so $X'' \to X$ is a semismooth resolution. Let E_1 denote the strict transform on Y'' of the exceptional curve E and let of the first blow up Arguing as in Case $4'.1$, one sees that inverse image of

E_1 is the union of two smooth rational curves F_1^1 and F_2^1 that intersect transversely precisely at one point of the double curve, and the inverse image of E_2 consists of 4 disjoint curves $F_2^1, \dots F_2^4$. All these curves pull back to -2 curves on the normalization of X'' and, up to relabeling, F_1^1, F_1^2, F_2^2 and F_1^2, F_2^3, F_2^4 form two disjoint A_3 configurations. So X'' is the minimal semiresolution of X. In the notation of [KSB88][def. 4.26], X'' is obtained by gluing two copies of (A, Δ) along Δ.

4.3. **Slc $\mathbb{Z}_2^k$ covers with n.c. surface Y.** Here we carry out the analysis of the singularities in the case that $Y = Y_1 \cup Y_2$ is n.c and $\pi\colon X \to Y$ is obtained by gluing two standard G-covers $\pi_i\colon X_i \to Y_i$. We denote by D_Y the double curve of Y and by C the image in Y of the double curve of X. We assume that the total branch divisor D is Cartier and that the components of D are "lines". Since $K + D$ is $\mathbb{Q}$-Cartier, through a point $y \in D_Y$ there are the same number of components of D on each side. The normalization of $\pi^{-1}(D_Y)$ can be computed by restricting first the cover to either component of Y. This fact imposes restrictions on the combinatorics of the two covers

Notation 4.6. The components of D through y are ordered so that all components on Y_1 come first. If D_Y is in the branch locus, then we take it as the 0^{th} component. The case $k.m$ means that there are k curve on each component of Y through y, in addition to the double locus. The case $2'.m$ means that two of the lines on the first side coincide, and $2''.m$ that both pairs of lines, on both components, coincide.

E stands for étale, and R for ramified, and refers to the double locus.

In listing the possible cases, in addition to the obvious symmetries, we have used the following remark. Assume that the curve D_Y is in the branch locus and that $g_0, \gamma_1, \dots \gamma_m$ is a $\mathbb{Z}_2$-basis of G. We can change the action of G on, say, X_2, by an automorphism of the form $g_0 \mapsto g_0$, $\gamma_i \mapsto \gamma_i + \varepsilon_i g_0$, where $\varepsilon_i = 0$ or 1. This corresponds to considering a different structure of G-cover on the map $X \to Y$, but of course the geometry of X is not changed. So, for instance, case $(R1.?)$ with $H = 4$ and relation 012 can be identified with case $(R1.1)$ with $H = 4$ and relation 12.

The singularities that we get here are non-normal, and as in [KSB88, Thm. 4.21, 4.23] they turn out to be either semismooth or degenerate cusps in the Gorenstein case and $\mathbb{Z}_2$-quotients of these otherwise. In the tables, the non Gorenstein cases are precisely those described as $\mathbb{Z}_2$-quotient of other cases. In order to decide whether a singularity is Gorenstein or not, we use several remarks: The singularity is Gorenstein if:

1a) the G-cover $X \to Y$ is obtained by restricting a cover $Z \to W$, with Z, W Gorenstein, $Y \subset W$ a Cartier divisor that intersects the branch locus of p properly;

2a) there is a map $X \to X'$, where X' is Gorenstein and the map is étale in codimension 1.

These remarks are enough to deal with the "E" cases: 1a), with $W = \mathbb{A}^3$, applies to all cases excepting (E2.1), (E2$'$.1) and (E2$''$.1a). These three cases are 2 : 1 covers, étale in codimension 1, of (E2.3), respectively (E2$'$.3), (E2$''$.3), hence they are Gorenstein by 2a).

We now examine some of the "R" cases. Let $\mathbb{Z}_2$ act on $\mathbb{A}^3$ by $(x, y, z) \mapsto (-x, -y, z)$, set $W := \mathbb{A}^3/\mathbb{Z}_2$ and let $p\colon \mathbb{A}_3 \to W$ be the quotient map. We can identify W with $\{x_2^2 = x_1x_3\} \subset \mathbb{A}^4$ and Y with $\{x_2 = 0\} \subset W$. The restriction of p to Y case corresponds to case (R0.1), which therefore is rational. Using this construction, one shows that cases (R1.1), (R2.4), (R2$'$.5), (R2$'$.9), (R2$''$.4) and (R2$''$.7) are Gorenstein by 1a). Case (R2.3) is the fiber product of (R0.1) and (E2.1), hence it is Gorenstein by applying 1a) twice. Cases (R2$'$.4) and (R2$''$.3) can be dealt with in the same way.

The singularity is not Gorenstein if:

1b) the discrepancies are not integers;

2b) the singularity is not semicanonical and the exceptional curves in the minimal semiresolution do not form a cycle.

Using condition 1b), one sees that cases (R2.8)–(R2.11) and (R2$'$.11)–(R2$'$.14) are not Gorenstein. Using condition 2b) one shows that the cases (R2$'$.10), (R2$''$.8), (R2$''$.9) and (R2$''$.10) are not Gorenstein, either.

Theorem 4.7. *The slc covers of the n.c union of two smooth surfaces are given in Tables 5–10.*

Table 5. D_Y not in the branch locus, D reduced

No.	$\lvert H\rvert$	Relations	Sing.	X^ν	$D_X^\nu \to D_X \to C$	$\widetilde{X}$	B
E0.1	1	none	n.c.	2(0.1)	$2\Delta \to \Delta \to \Delta$	n.c.	B
E1.1	2	12	n.c.	2(1.1)	$2\Delta \to \Delta \xrightarrow{2:1} \Delta$	n.c.	
E2.1	4	12 34	deg.cusp(4)	$2(2.2) \sqcup 2(2.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Delta$	n.c.	B
E2.2	4	13 24	deg.cusp(2)	$(2.1) \sqcup (2.1)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \xrightarrow{(2,2:1)} \Delta$	n.c.	
E2.3	2	12 13 14	deg.cusp(2)	$(2.2) \sqcup (2.2)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	

Table 6. D_Y not in the branch locus, $D|_{Y_1}$ non reduced, $D|_{Y_2}$ reduced

No.	$\lvert H\rvert$	Relations	Singularity	X^ν	$D_X^\nu \to D_X \to C$	$\widetilde{X}$	B
E2′.1	4	12 34	deg.cusp(6)	$4(0.1) \sqcup 2(2.2)$	$4\Gamma_2 \sqcup \Gamma_2 \to \Gamma_6 \to \Gamma_2$	n.c.	B
E2′.2	4	13 24	deg.cusp(3)	$2(1.1) \sqcup (2.1)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{(1,2,2:1)} \Gamma_2$	n.c.	
E2′.3	2	12 13 14	deg.cusp(3)	$2(0.1) \sqcup (2.2)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \to \Gamma_2$	n.c.	

Table 7. D_Y not in the branch locus, $D|_{Y_1}$ and $D|_{Y_2}$ non reduced

No.	$\lvert H\rvert$	Relations	Singularity	X^ν	$D_X^\nu \to D_X \to C$	$\widetilde{X}$	B
E2″.1	4	12 34	deg.cusp(8)	$4(0.1) \sqcup 4(0.1)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \to \Gamma_3$	n.c.	
E2″.2	4	13 24	deg.cusp(4)	$2(1.1) \sqcup 2(1.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{(1,2,2,1:1)} \Gamma_3$	n.c	
E2″.3	2	12 13 14	deg.cusp(4)	$2(0.1) \sqcup 2(0.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Gamma_3$	n.c.	

Table 8. D_Y in the branch locus, D reduced

No.	$\lvert H\rvert$	Relations	Singularity	X^ν	$D_X^\nu \to D_X \to C$	$\widetilde{X}$	B
R0.1	2	none	n.c.	$(1.1) \sqcup (1.1)$	$\Delta \sqcup \Delta \to \Delta \to \Delta$	n.c.	B
R1.1	4	12	n.c.	$(2.1) \sqcup (2.1)$	$\Delta \sqcup \Delta \to \Delta \overset{(2:1)}{\to} \Delta$	n.c.	B
R1.2	2	01 02	[KSB88, 4.23(iii)]	$(2.2) \sqcup (2.2)$	$\Delta \sqcup \Delta \to \Delta \to \Delta$	n.c.	B
R2.1	8	012 034	deg.cusp(4)	$2(3.2) \sqcup 2(3.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Delta$	n.c.	
R2.2	8	12 034	deg.cusp(8)	$2(3.3) \sqcup 2(3.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Delta$	n.c.	
R2.3	8	12 34	deg.cusp(12)	$2(3.3) \sqcup 2(3.3)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Delta$	n.c.	
R2.4	8	13 24	deg.cusp(2)	$(3.1) \sqcup (3.1)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \xrightarrow{(2,2:1)} \Delta$	n.c.	
R2.5	4	012 013 14	deg.cusp(2)	$(3.2) \sqcup (3.2)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	
R2.6	4	012 13 14	deg.cusp(4)	$(3.2) \sqcup (3.3)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	
R2.7	4	12 13 14	deg.cusp(6)	$(3.3) \sqcup (3.3)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	
R2.8	4	01 02 034	$(\text{R2.2})/\mathbb{Z}_2$	$2(3.4) \sqcup (3.2)$	$2\Delta \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	B
R2.9	4	01 02 34	$(\text{R2.3})/\mathbb{Z}_2$	$2(3.4) \sqcup (3.3)$	$2\Delta \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	
R2.10	4	01 03 24	$(\text{R2.4})/\mathbb{Z}_2$	$(3.3) \sqcup (3.3)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	n.c.	
R2.11	2	01 02 03 04	$(\text{R2.7})/\mathbb{Z}_2$	$(3.4) \sqcup (3.4)$	$\Delta \sqcup \Delta \to \Delta$	n.c.	

TABLE 9. D_Y in the branch locus, $D|_{Y_1}$ non reduced, $D|_{Y_2}$ reduced

No.	$\|H\|$	Relations	Singularity	X^ν	$D_X^\nu \to D_X \to C$	$\widetilde{X}$	B
R2′.1	8	012 034	deg.cusp(6)	$4(2.2) \sqcup 2(3.2)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \to \Gamma_2$	n.c.	
R2′.2	8	012 34	deg.cusp(10)	$4(2.2) \sqcup 2(3.3)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \to \Gamma_2$	n.c.	
R2′.3	8	12 034	deg.cusp(6)	$4(1.1) \sqcup 2(3.2)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \xrightarrow{(2,2,1\dots1:1)} \Gamma_2$	n.c.	
R2′.4	8	12 34	deg.cusp(10)	$4(1.1) \sqcup 2(3.3)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \xrightarrow{(2,2,1\dots1:1)} \Gamma_2$	n.c.	
R2′.5	8	13 24	deg.cusp(3)	$2(2.1) \sqcup (3.1)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{(2,2,2:1)} \Gamma_2$	nc.	
R2′.6	4	012 013 14	deg.cusp(3)	$2(2.2) \sqcup (3.2)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \to \Gamma_2$	n.c.	
R2′.7	4	012 13 14	deg.cusp(5)	$2(2.2) \sqcup (3.3)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \to \Gamma_2$	n.c.	
R2′.8	4	12 013 14	deg.cusp(3)	$2(1.1) \sqcup (3.2)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{(2,1,1:1)} \Gamma_2$	n.c.	
R2′.9	4	12 13 14	deg.cusp(5)	$2(1.1) \sqcup (3.3)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{(2,1,1:1)} \Gamma_2$	nc.	B
R2′.10	4	01 02 034	$(\text{R2}'.3)/\mathbb{Z}_2$	$2(1.1) \sqcup (3.2)$	$2\Gamma_2 \to \Gamma_3 \xrightarrow{(2,1,1:1)} \Gamma_2$	pinch	B
R2′.11	4	01 02 34	$(\text{R2}'.4)/\mathbb{Z}_2$	$2(1.1) \sqcup (3.3)$	$2\Gamma_2 \to \Gamma_3 \xrightarrow{(2,1,1:1)} \Gamma_2$	pinch	
R2′.12	4	012 03 04	$(\text{R2}'.2)/\mathbb{Z}_2$	$2(2.2) \sqcup (3.4)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_3 \to \Gamma_2$	n.c.	B
R2′.13	4	12 03 04	$(\text{R2}'.4)/\mathbb{Z}_2$	$2(1.1) \sqcup (3.4)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{(2,1,1:1)} \Gamma_2$	n.c.	
R2′.14	2	01 02 03 04	$(\text{R2}'.11)/\mathbb{Z}_2$	$(1.1) \sqcup (3.4)$	$\Gamma_2 \sqcup \Delta \to \Gamma_2 \xrightarrow{(2,1:1)} \Gamma_2$	pinch	

TABLE 10. D_Y in the branch locus, $D|_{Y_1}$ and $D|_{Y_2}$ non reduced

No.	$\|H\|$	Relations	Singularity	X^ν	$D_X^\nu \to D_X \to C$	$\widetilde{X}$	B
R2″.1	8	012 034	deg.cusp(8)	$4(2.2) \sqcup 4(2.2)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \to \Gamma_3$	n.c.	
R2″.2	8	012 34	deg.cusp(8)	$4(2.2) \sqcup 4(1.1)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \xrightarrow{(1\dots1,2,2:1)} \Gamma_3$	nc.	
R2″.3	8	12 34	deg.cusp(4)	$4(1.1) \sqcup 4(1.1)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \xrightarrow{(2,2,1,1,1,1,2,2:1} \Gamma_3$	n.c.	
R2″.4	8	13 24	deg.cusp(4)	$2(2.1) \sqcup 2(2.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{(2,2,2,2:1)} \Gamma_3$	n.c.	
R2″.5	4	012 013 14	deg.cusp(4)	$2(2.2) \sqcup 2(2.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \to \Gamma_3$	n.c.	
R2″.6	4	012 13 14	deg.cusp(4)	$2(2.2) \sqcup 2(2.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \xrightarrow{(1,\dots,1,2:1)} \Gamma_3$	n.c.	B
R2″.7	4	12 13 14	deg.cusp(4)	$2(1.1) \sqcup 2(1.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{(2,2,2,2:1)} \Gamma_3$	n.c.	B
R2″.8	4	012 03 04	$(\text{R2}''.2)/\mathbb{Z}_2$	$2(2.2) \sqcup 2(1.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Gamma_3$	pinch	B
R2″.9	4	12 03 04	$(\text{R2}''.3)/\mathbb{Z}_2$	$2(1.1) \sqcup 2(1.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{(1,1,1,2:1)} \Gamma_3$	pinch	
R2″.10	2	01 02 03 04	$(\text{R2}''.7)/\mathbb{Z}_2$	$(1.1) \sqcup (1.1)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \to \Gamma_3$	pinch	

5. Compactifying moduli of abelian Galois covers

5.1. **One-parameter degenerations of abelian covers.** Consider one of the degenerations of Burniat arrangements we constructed in Section 2.2. Recall that in each case we did one or several blowups at points or lines, followed by a contraction given by big and nef $K_{\mathcal{X}} + \mathcal{D}$. What happens for the corresponding $\mathbb{Z}_2^2$ covers?

Every time we blow up a point or a line contained in the divisors D_{h_i}, the exceptional divisor gets a label $h = \sum h_i$. If $h \neq 0$ then the whole exceptional divisor E appears in D_h with multiplicity 1. After the base change $t = s^2$, the exceptional divisor appears in D_h with multiplicity 2. The corresponding G-cover is not normal. For the normalization, the new divisor is $D_h' = D_h - 2E$, and only the double curve appears in D_h. This explains the coloring rules for the double locus.

5.2. **General existence theorem.**

Theorem 5.1. *Let $\overline{\mathrm{M}}$ be the compactified moduli of Campedelli, Uniform Hyperplane, or Burniat arrangements constructed in Section 2. Then there exists a finite $\mathbb{Z}_2^n$ cover $\overline{\mathrm{M}}' \to \overline{\mathrm{M}}$ (where $n = 2^k - 1$ for Uniform Hyperplane and $n = 9$ for Burniat) together with a flat projective morphism $\mathcal{X}' \to \overline{\mathrm{M}}'$ and a finite morphism $\pi' : \mathcal{X}' \to (\mathcal{Y}', \mathcal{D}')$, where $(\mathcal{Y}', \mathcal{D}') = (\mathcal{Y}, \mathcal{D}) \times_{\overline{\mathrm{M}}} \overline{\mathrm{M}}'$, such that:*

1. *On geometric fibers $\pi'_s : X'_s \to Y'_s$ is a $\mathbb{Z}_n^2$ Galois cover with the ramification divisor D'_s.*
2. *Every geometric fiber X'_s of $\mathcal{X}' \to \overline{\mathrm{M}}'$ is a stable surface of index 2, i.e. X'_s is slc and $2K_{X'}$ is an ample Cartier divisor.*
3. *Two geometric fibers X'_{s_1}, X'_{s_2} are isomorphic iff s_1, s_2 have the same image in $\overline{\mathrm{M}}$.*

We will call $\overline{\mathrm{M}}$ the *coarse moduli space of stable Campedelli (resp. Uniform Line Cover or Burniat) surfaces.*

Proof. The space $\overline{\mathrm{M}}$ comes with a projective flat family of pairs $(Y, D = \frac{1}{2}\sum D_h)$ such that all geometric fibers are distinct. For each fixed fiber, we can construct from it the G-cover X. But rather than trying to organize these fibers in a family, we proceed more directly. Recall that in [Ale08] the family over $\overline{\mathrm{M}}$ of slc hyperplane arrangements was constructed as follows.

One starts with the moduli $\overline{\mathrm{M}}$ of stable toric varieties over the weighted grassmannian and its universal family $\widetilde{Y} \to \overline{\mathrm{M}} \times \mathrm{Gr}_\beta(r,n)$. Let $U \to \mathrm{Gr}_\beta(r,n)$ be the universal family of pairs $(\mathbb{P}V, \sum_{i=1}^n b_i B_i)$. Then the family of slc hyperplane arrangements over $\overline{\mathrm{M}}$ is the GIT quotient

$$\mathcal{Y} = (\widetilde{Y} \times_{\mathrm{Gr}(r,n)} U)//_\beta T, \qquad \text{where } \mathbb{G}_m^n / \operatorname{diag}(\mathbb{G}_m).$$

In our case, we have $\overline{\mathrm{M}}$, which is the moduli space of stable toric varieties over $\mathrm{Gr} = \mathrm{Gr}_{(\frac{1}{2},\dots,\frac{1}{2})}(r,n)$ for Uniform Hyperplane, and over $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{Bur}}$ for Burniat. Over Gr, we have the universal family U of pairs $(\mathbb{P}V, \frac{1}{2}\sum_{i=1}^n D_h)$, resp. of pairs $(\mathrm{Bl}_3 \mathbb{P}V, \frac{1}{2}\sum_{i=0}^3 (A_i + B_i + C_i))$.

Now, over Gr, we want to construct the universal family of G-covers (some of which may be very singular, but the GIT semistable locus will be equal precisely to the set of lc points). For this family, we need to find the sheaves L_χ which are the half-sums of some of the divisors D_h.

The line bundles exist fiberwise, since on Σ the divisors $A+B$, $B+C$, $C+A$ are uniquely divisible by 2 in $\mathrm{Pic}(\Sigma)$. But we need them to exist globally in a family, and be $\mathbb{G}_m^n$-linearized so that the weights are in the character group $\mathbb{Z}^n$ and not in $\frac{1}{2}\mathbb{Z}^n$. Note that the sheaves $\mathcal{O}(A+B), \mathcal{O}(B+C), \mathcal{O}(C+A)$ are $\mathbb{G}_m^n$-linearized.

After making a base change $T \to T$ squaring every coordinate $\mathbb{G}_m$, the sheaves L_χ exist, and they come with a canonical linearization. The result of this base change is the base changes $\overline{\mathrm{M}}' \to \overline{\mathrm{M}}$, $\mathrm{Gr}' \to \mathrm{Gr}$, and we have the universal family over $U' \to \mathrm{Gr}'$.

We now define $\mathcal{X}'$ as the GIT quotient

$$\mathcal{X}' = \big((\operatorname{Spec}_{\widetilde{Y} \times_{\mathrm{Gr}(r,n)} U'} \oplus_{h \in G^*} L_\chi^{-1}) \times_{\mathrm{Gr}} U'\big)//T$$

[Ale08, Thm.6.6] for Uniform Hyperplane, resp. Lemma 2.8 for Burniat give a family of slc surfaces over $\overline{\mathrm{M}}'$. It comes with a natural finite morphism to $\mathcal{Y}' = \mathcal{Y} \times_{\overline{\mathrm{M}}} \overline{\mathrm{M}}'$. By [Ale08, Thm.1.1(2)] two pairs $(Y, \frac{1}{2}\sum D_h)_s$ are isomorphic iff the images of s are the same in $\overline{\mathrm{M}}$, and so the same is true for the covers X_s. □

5.3. **Campedelli surfaces.** Here we plug in the results of Sections 2.1, 4.2, 5.2, and do a little enumeration.

Theorem 5.2. *The coarse moduli space of stable Campedelli surfaces is $\overline{\mathrm{M}} = (\mathbb{P}^2)^7 // \mathrm{PGL}(3)$, a normal 6-dimensional variety. There are two boundary divisors up to the action of $\mathrm{GL}(3, \mathbb{F}_2)$. They correspond to the cases when three lines D_{g_i} pass through the same point and either g_i are independent (A_1 singularity, case 3.1 of Table 1 in Section 4) or $\sum g_i = 0$ ($\frac{1}{4}(1,1)$ singularity, case 3.2).*

Remark 5.3. It is a straightforward but tedious exercise to list the boundary data of higher codimensions. For example in codimension 2, either 2 triples of lines pass through a common point (which gives many cases depending on whether 5 or 6 lines participate, and also whether some triples add up to zero in $\mathbb{Z}_2^3$), or two lines coincide (one case, with the singularities $2'.1$, $3'.1$, $3'.2$).

Recall that we listed all the singularities appearing on stable Campedelli surfaces in Tables 2, 3, 4.

Proposition 5.4. *Stable Campedelli surfaces have 1,2, or 4 irreducible components.*

Proof. The case of 8 components never occurs. This would mean that after applying the normalization procedure the subgroup of $\mathbb{Z}_2^3$ generated by the new labels is trivial. This means that all the lines "cancel" each other, i.e. $D_h = D_{h'}$ with $h + h' = 0$. This does not happen since 7 is odd and h are distinct.

The case of four irreducible components occurs when $B_{100} = B_{011}$, $B_{010} = B_{101}$, $B_{001} = B_{110}$, where we use the natural labels for the nonzero elements of $\mathbb{Z}_2^3$. In this case the normalization has 4 components, each of them a double cover of $\mathbb{P}^2$ ramified in 4 lines corresponding to $g = 111$. Each component is a del Pezzo surface of degree 2 with $6A_1$ singularities. It is easy to see that up to the action of $\mathrm{GL}(3, \mathbb{F}_2)$ this is the only case with 4 irreducible components.

If we split one of the double lines then the cover has 2 components. Each of them is a del Pezzo of degree 1 with $6A_1$ singularities. □

5.4. **Burniat surfaces.** Here we plug in the results of Sections 2.2, 2.3, 4.2, 4, 5.2, and enumerate the boundary divisors.

Theorem 5.5. *The coarse moduli space of stable Burniat surfaces is an irreducible 4-dimensional variety. The boundary divisors, up to the action of the symmetry group, are:*

(1) *Case 1 of Section 2.2.*
(2) *Case 2.*
(3) *Case 6.*
(4) *Case 8.*
(5) *Case 9*
(6) $Y = \Sigma$ *and two lines in the same pencil coincide, for example* $A_1 = A_2$.
(7) $Y = \Sigma$ *and* A_1 *degenerates to* $A_0 + C_3$.
(8) $Y = \Sigma$ *and three lines in different pencils, e.g.* A_1, B_1, C_1*, pass through the same point.*

Again, the proof is immediate, as the sum of the above sections. Finally, for each of the above boundary divisors, we describe a general stable Burniat surface.

(1) *Case 1.* In the general case, namely when all the lines are distinct, each component is a smooth bielliptic surface (so $K^2 = p_g = 0$, $q = 1$) and the Albanese pencil is the pull back of the ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ that contains two pairs of branch lines in the same ramification divisor D_h. Two components are glued transversally along a smooth elliptic curve. The three components of the double curve of X meet at one point, which is a degenerate cusp of X. When two lines coincide we get degenerate elliptic surfaces.

Another description, useful in understanding the degenerations, is as follows. For the general case, consider three elliptic curves E_1, E_2 and E_3, and on each E_i a translation τ_i by a point of order 2 and a rational involution σ_i. Let σ_i' be the involution induced by σ_i on $E_i' := E_i/\tau_i$. Take $X_i := (E_{i+1} \times E_{i+2}')/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on E_{i+1} via τ_{i+1} and on E_{i+2}' via σ_{i+2}' (the index i varies in $\mathbb{Z}_3$). The surfaces X_i and X_{i+1} are glued along a curve isomorphic to E_{i+2}', which on X_i is a fiber of the Albanese pencil $X_i \to E_{i+1}'$ and on X_{i+1} is half of a fiber of the rational pencil $X_{i+1} \to E_i'/\sigma_i' = \mathbb{P}^1$.

Letting two lines in the same ramification divisor coincide corresponds to degenerating one of the E_i to a cycle of two rational curves. Letting two lines that are in different ramification divisors on one component coincide corresponds to degenerating one of the E_i to a nodal rational curve. Up to three degenerations of this type can occur at the same time.

This surface appears very nicely as a degeneration of Burniat surface in the form given by Inoue [Ino94], with the parameter $\lambda \to 0$ or ∞. It is easy to compute that $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$.

(2) *Case 2.* In the general case, the three components X_1, X_2 and X_3, are degenerate Enriques surfaces. The surfaces X_i meet transversally at one point P_0 which is smooth for all of them, so X has a degenerate cusp there. Two components X_i and X_{i+1} are glued along a rational curve with a node P_{i+2}. At P_{i+2} there is additional gluing and the surface such that P_{i+2} lies on 3 lines in the same branch divisor is not S_2 there.

When one of the components of, say, X_1, splits into the union of two surfaces Z_1, Z_2, then each Z_i is a degenerate Del Pezzo surface with $K^2 = 1$. The surfaces Z_1 and Z_2 are glued along a rational curve with a node, and also in this case one of the components is not S_2 at the node. The point where the 4 components meet is a degenerate cusp. When X has 5 or 6 components, the situation is similar.

(3) *Case 6.* Each component X_i is a degenerate properly elliptic surface with $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$. The elliptic fibration is given by $|2K|$ and it is the pullback of the ruling of $\mathbb{F}_1$. The two components are glued

along the union of two smooth rational curves meeting at two points P_1, P_2, where there is an additional gluing. Each component is not S_2 at one of the points P_i (the one lying on the lines in D_b). A somewhat nontrivial computation shows that $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$.

(4) *Case 8.* The component X_1 which is the cover of the blow up Y_1 of Σ at one point has $K^2 = 2$, $h^1(\mathcal{O}) = 1$, $h^2(\mathcal{O}) = 0$. It is not normal: it has n.c. singularities along the two double lines in the branch locus. The bicanonical system is free and maps X_1 onto a smooth quadric in $\mathbb{P}^3$. (This surface cannot be smoothed to a surface of general type because it has $\chi = 0$.) The second component X_2 is a degenerate Enriques surface. Both components are S_2 and they are glued along the union of two rational curves meeting transversally at two points.

(5) *Case 9.* The surface X_1 which is the double cover of $Y_1 = \mathbb{P}^1 \times \mathbb{P}^1$ is a degenerate del Pezzo surface with $K^2 = 1$, the quotient $\mathbb{P}^1 \times \mathbb{P}^1$ by $\mathbb{Z}_2$ acting as $(x, y) \mapsto (-x, -y)$. The second component X_2 is a degenerate Enriques surface. The two surfaces are glued along the union of 4 smooth rational curves, all passing through 3 points at which there is extra gluing, so that neither surface is S_2 there. If we let two of the pairs of lines in the same ramification divisor coincide on Y_2, then X_2 becomes reducible: it is the union of two quadric cones glued along the union of two plane sections not passing through the vertex. If all three pairs of lines get to coincide, then X_2 is the union of 4 planes, glued along 6 rational curves. These 6 curves fall into 3 pairs, according to the ramification divisor of the image curve, and there are 3 points Q_1, Q_2, Q_3 on X_2 such that each Q_i lies on two pairs of curves. Two curves in the same branch divisor D_h also have an extra intersection at one of the P_i. A nontrivial computation shows that $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$.

(6) $Y = \Sigma$ *and two lines in the same pencil coincide, for example* $A_1 = A_2$. The surface becomes non-normal, with the singularities of type 2'.2 and 3'.3. The normalization has a fibration over $\mathbb{P}^1$ with the fiber a curve of genus 3.

(7) $Y = \Sigma$ *and* A_1 *degenerates to* $A_0 + C_3$. This is similar to the previous case, but the surface has singularities of types $2'.1$, $2'.2$, $3'.2$, $3'.3$, and $4''.10$.

(8) $Y = \Sigma$ *and three lines in different pencils, for example* A_1, B_1, C_1 *pass through the same point.* The surface acquires a $\frac{1}{4}(1, 1)$ singularity.

Remark 5.6. It is instructive to compute that for all the surfaces above one has $\chi(\mathcal{O}_X) = 1$, as it should be since they are flat limits of smooth Burniat surfaces.

Remark 5.7. Although the space $\overline{\mathrm{M}}_{\mathrm{Bur}}$ which we constructed is irreducible, in the larger space of stable surfaces there are definitely other irreducible components intersecting $\overline{\mathrm{M}}_{\mathrm{Bur}}$. For example, in case 9 the pairs of lines on $\mathbb{P}^2$ can be deformed to conics D_a, D_b, D_c tangent to the double locus. Similarly, the three divisors of type $(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ can be smoothed, keeping them tangent to the double locus. Since the induced $\mathbb{Z}_2^2$ covers of the double curve $\mathbb{P}^1$ have the same normalization, the covers can be glued. This gives a family of dimension 12. Many of the other degenerations produce other irreducible components in the moduli of stable surfaces.

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Appendix: Cremona transformation for stable h.a.s

Although this is not essential for the proofs of the main results of this paper, below we illustrate how the Cremona transformation acts on the stable hyperplane arrangements. On the left and right are the degenerate hyperplane arrangements, the limits of $(\mathbb{P}^2, \frac{1}{2}\sum_{i=0}^{2}(A_i+B_i+C_i))$ and of $(\mathbb{P}^2, \frac{1}{2}\sum_{i=1}^{3}(A_i+B_i+C_i))$ respectively. In the middle is the degeneration of $(\Sigma, \frac{1}{2}\sum_{i=0}^{3}(A_i+B_i+C_i))$.

FIGURE 1. Case 1

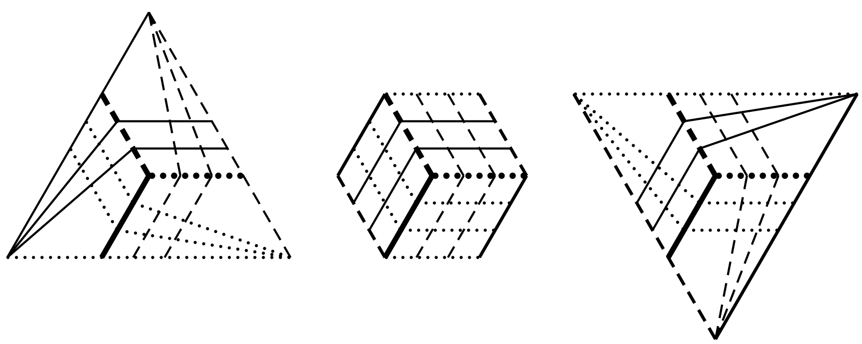


Figure 1 is the generic situation. For a non-generic configuration of curves on the stable Burniat surface (for example, when on the first $\mathbb{P}^2$ the line A_1 degenerates to A_0 faster than the line A_2) one of the $\mathbb{F}_1$ components may further split into $(\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{P}^2$. The stable Burniat surface remains the same, only the configuration of curves changes, remaining slc.

So there are many more types of non-normal surfaces appearing as limits of $\mathbb{P}^2$ than as limits of Σ.

FIGURE 2. Case 2

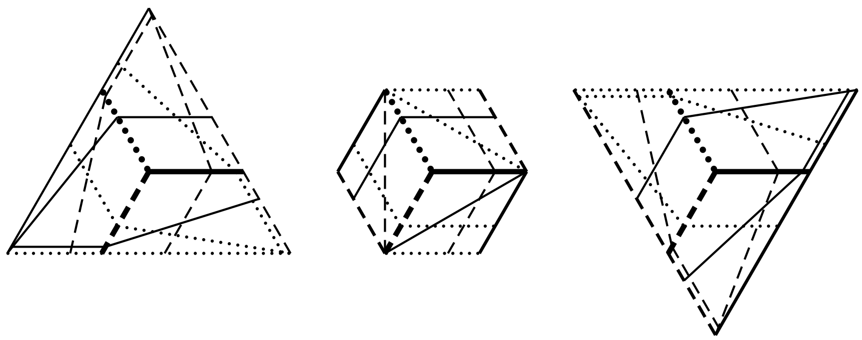


FIGURE 3. Case 5

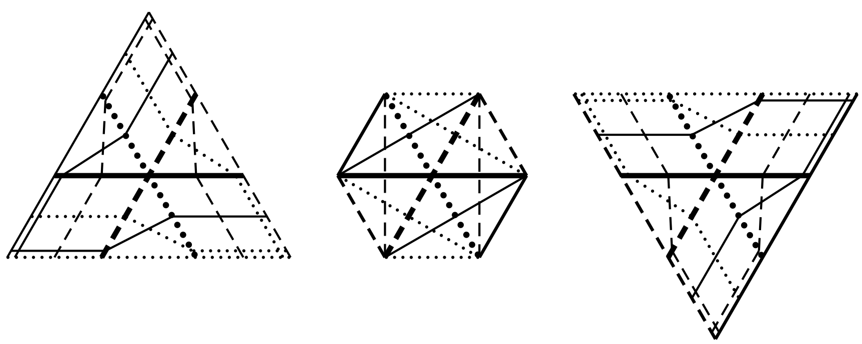
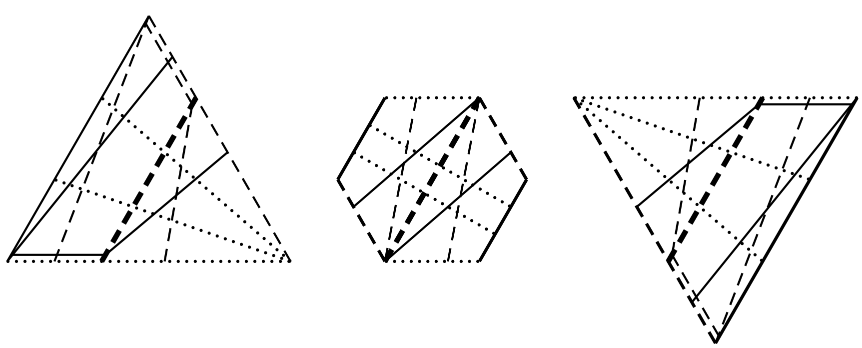


FIGURE 4. Case 6.



In the non-toric cases 8, 9, 10 the first irreducible component of the limit hyperplane arrangement is $\mathrm{Bl}_1 \mathbb{P}^2$, and the Cremona involution acts on it in the usual way, transforming it to another $\mathrm{Bl}_1 \mathbb{P}^2$.