April 1996.

Ph.D. Preliminary Exam: Real Analysis

(1) Let $f(x)$ be a continuous function on $(-\infty, \infty)$. Show that

$$\lim_{h\to 0}\frac{1}{h}\int_0^1 (f(x+h)-f(x))dx = f(1)-f(0).$$

(2) (a) Use the ratio test to show the series $\sum_{n=1}^{\infty}\frac{n}{2^n}$ converges.

(b) Use the power series $f(x)=\sum_{n=1}^{\infty} nx^{n-1}$ to compute $\sum_{n=1}^{\infty}\frac{n}{2^n}$.

(3) Let $A=(a_{ij})$ be an $n\times n$ matrix. Then, for any column vector $x\in R^n$, the product $Ax\in R^n$. Define

$$||A|| = \sup_{x\neq 0}\frac{|Ax|}{|x|},$$

where $|x|=\sqrt{x_1^2+\cdots+x_n^2}$.

(a) Show that $||\cdot||$ defines a norm on $R^{n\times n}$.

(b) Show that if $||A||<1$, then $A: R^n\to R^n$ is contractive mapping. Therefore the equation $Ax=x$ has a unique solution.

(4) Let $f(x)$ be a monotone function on $[0,1]$. Show that $\{x : f \text{ is discontinuous at } x\}$ is at most countable.

(5) Let $\{f_n(x)\}$ be a sequence of Borel measurable functions. Show that $\limsup_{n\to\infty} f_n(x)$ is also Borel measurable.

(6) Let $f(x)$ be a Lebesgue integrable function on $[-\delta, 1+\delta]$ for some $\delta>0$. Show that

$$\lim_{h\to 0}\int_0^1 |f(x+h)-f(x)|dx = 0.$$

(7) Let $\{f_n(x)\}$ denote a sequence of nonnegative functions. Suppose that $\int f_n(x)dx\to 0$ as $n\to\infty$. Show that $f_n\to 0$ in measure.

(8) Let $\{x_n\}$ and $\{a_n\}$ be two sequences. Suppose for all $\{x_n\}\in l^1$, $\sum_{n=1}^{\infty} a_n x_n$ converges. Prove that $\{a_n\}\in l^{\infty}$, i.e., $\sup_n |a_n|<\infty$.

(9) If μ is a positive measure and E_1, E_2, ... are measurable sets satisfying

$$\sum_{n=1}^{\infty}\mu(E_n)<\infty,$$

prove that

$$\lim_{n\to\infty}\mu\left(E_1\bigcup\cdots\bigcup E_n\right)=\mu\left(\bigcup_{n=1}^{\infty}E_n\right).$$