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Real Analysis Preliminary Examination
May, 1993

1. Let $\{f_n\}$ be a sequence of differentiable functions on $(0, 1)$, converging uniformly to a differentiable function f.

   (a) Give an example to show that the derivatives f_n' need not converge uniformly to f'.

   (b) Show that if f_n' converges uniformly to some function g, then $f' = g$.

2. State and prove a Mean Value Theorem for a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $n > 1$. Be sure to include all necessary hypotheses for the domain D and the function f. (You may use without proof the mean value theorem for $g : \mathbb{R} \to \mathbb{R}$).

3. Prove that a continuous real-valued function on a compact metric space is uniformly continuous.

4. (a) Define the Lebesgue mesaure of a measurable set $A \subseteq \mathbb{R}$.

   (b) Prove, using the definition, that the Lebesgue measure of $[0, 1]$ equals 1.

5. Let $\phi(x)$ be a continuous function on $\mathbb{R}$, with $\int_{-\infty}^{\infty} \phi(x)dx = 1$ and $\phi(x) = 0$ $\forall |x| > 1$. Prove that for any continuous function f on $\mathbb{R}$, $\int_{-\infty}^{\infty} f(x)n\phi(nx)dx \to f(0)$ as $n \to \infty$.

6. A probability measure μ on $\mathbb{R}$ is a positive Borel measure μ with $\mu(\mathbb{R}) = 1$. If μ, ν are probability measures on $\mathbb{R}$, then so is $\mu * \nu$, defined by $(\mu * \nu)(A) = \iint \chi_A(s+t)d\mu(s)d\nu(t)$.

   (a) Verify that $\mu * \nu$ is countably additive.

   (b) Verify carefully that $\mu * \nu = \nu * \mu$.

7. Assume that all measures are positive and finite.

   (a) Prove that if μ_1 and μ_2 are each singular with respect to ν, then so is $\mu_1 + \mu_2$.

   (b) Suppose that μ is singular with respect to ν. Compute the Radon-Nikodym derivative of μ with respect to $\mu + \nu$.

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(c) Suppose that μ is absolutely continuous with respect to ν. In terms of the Radon-Nikodym derivative of μ with respect to ν, compute the Radon-Nikodym derivative of μ with respect to $\mu + \nu$.

8. Let X be a Banach space.

   (a) Prove that the canonical mapping $X \to X^{**}$ is an isometry.

   (b) Give an example of a Banach space X which is not reflexive, and verify your example by computing X^*, X^{**}.

9. Prove that the Banach space ℓ^1 is separable, and that the Banach space ℓ^∞ is not separable.