1. Prove that every real-valued continuous function on $[0, \pi]$ can be uniformly approximated with trigonometric polynomials of the form $a_0 + a_1 \cos(x) + b_1 \sin(x) + \ldots + a_n \cos(nx) + b_n \sin(nx)$, $a_i, b_i \in \mathbb{R}$. Also, prove the same fact for real-valued continuous functions $f$ on $[-\pi, \pi]$ for which $f(-\pi) = f(\pi)$.

2. Let $S_n(x) := \sum_{k=1}^{n} \frac{\sin(kx)}{k^2}$ for $x \in [0, 2\pi]$, $n \geq 1$. Show that the following limit exists for every Lebesgue integrable function $f$ on $[0, 2\pi]$:

$$
\lim_{n \to \infty} \int_0^{2\pi} S_n(x)f(x)dx.
$$

3. Let $\varphi(x) = 2x - x^2$, $x \in \mathbb{R}$. For every Lebesgue measurable set $E \subset \mathbb{R}$ let $\mu_\varphi(E) = \mu(\varphi^{-1}(E))$, where $\mu$ is the Lebesgue measure. Show that $\mu_\varphi$ is a measure which is absolutely continuous with respect to $\mu$ and compute the Radon-Nikodym derivative of $\mu_\varphi$ with respect to $\mu$.

4. Let $A$ and $B$ be measurable subsets of the reals and let $\mu$ be the Lebesgue measure. Suppose that $0 < \mu(A), \mu(B) < \infty$. Prove the identity

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_B(t) \chi_{A+x}(t) d\mu(t) d\mu(x) = \mu(A) \mu(B),$$

and show that there exists an $x_0$ such that $\mu(B \cap (A + x_0)) > 0.$
5. Give an example of a map which is continuous at all the points of the Cantor set and discontinuous at all the other points in $[0,1]$.

6. If $f$ is a differentiable mapping of a convex open set $E \subset \mathbb{R}^2$ ($n > 1$) into $\mathbb{R}$, and $\frac{\partial f}{\partial x}(x,y) = 0$ for every $(x,y) \in E$, prove that $f$ depends only of the variable $y$. Show that this is not true anymore if for instance $E = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$.

7. State the implicit function theorem. Prove this theorem for the case of a linear map.

8. Let $M$ be the collection of continuous maps $f$ on $[0,1]$ with the property:

$$\int_0^{1/4} f(t)\,dt - \int_{3/4}^1 f(t)\,dt = 1.$$ 

Show that $M$ is a closed (with respect to the usual norm on continuous functions $\|f\|_{\infty} = \sup\{f(x) : x \in [0,1]\}$) and convex set, but there is no element in $M$ of minimal norm (i.e. there exist no $f \in M$ such that $\|f\|_{\infty} = \inf\{\|g\|_{\infty} : g \in M\}$).

9. Consider the continuous function $f : (0,\infty) \to \mathbb{R}$ with the property $\lim_{n \to \infty} f(nx) = 0$ for every $x \in [1,2]$. Use the Baire category theorem to show that $\lim_{x \to \infty} f(x) = 0$. 