Real Analysis Prelimary Exam
Fall Semester 1998, 24 Aug., 1998

Do as many of these problems as you can. Answers to questions should be proofs or counterexamples. Examples and counterexamples should be supplied with explanations of why they have the appropriate properties.

1.) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (x^3 + 2xy + y^2, x^2 + y).$$

Show that there is a function g, defined on a neighborhood of $(4,2)$, such that $g(4,2) = (1,1)$ and $f \circ g(x,y) = (x,y)$ for all (x,y) in that neighborhood. Give an affine function that best approximates g at $(4,2)$.

2.) Let $S \subset \mathbb{R}$ and assume every continuous function on S is uniformly continuous. Must S be compact?

3.) Give an example of a sequence of differentiable functions which converges uniformly to a nondifferentiable function.

4.) Let $f(x)$ be a monotone function on $[0,1]$. Show that

$$\{x | f(x) \text{ is discontinuous at } x\}$$

is at most countable.

5.) Is there a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous precisely at the rationals?

6.) Give (with proof) an explicit representation for the bounded linear functionals on $\ell^p, 1 < p < \infty$.

7.) Prove or give a counterexample: (Assume the functions are defined on a σ-finite measure space.)

a) If $f_n \to f$ in measure, then some subsequence of $\{f_n\}$ converges to f a.e.

b) If $f_n \to f$ in measure, then $f_n \to f$ in L^1

8.) Suppose that $f(x)$ and $xf(x)$ are Lebesgue integrable functions on $\mathbb{R}$. Show that the function

$$g(t) = \int_{\mathbb{R}} f(x) \cos(tx) dx$$

is differentiable.

9.) Let (X, M, μ) be a measure space and g a nonnegative measurable function on X.

Show that

$$\nu(E) = \int_E g \, d\mu$$

defines a measure on X and that for any nonnegative measurable function f.

$$\int_E f \, d\nu = \int_E fg \, d\mu.$$

10.) Let (X, M, μ) be a σ-finite measure space and $f : X \to \mathbb{R}$ a non-negative measurable function. Let

$$A = \{(x, t) \in X \times \mathbb{R} | \ 0 \leq t < f(x)\}.$$

Prove that A is measurable with respect to the product measure $\mu \times \lambda$ on $X \times \mathbb{R}$ (λ is Lebesgue measure) and that

$$\mu \times \lambda \, (A) = \int_X f \, d\mu.$$