Real.

Ph.D. Preliminary Examination
Real Analysis

Do any 8 problems.

The space L^p refers to the space of Lebesgue measurable functions on $[0, 1]$ satisfying

$$\int_0^1 |f(x)|^p \, dx < \infty.$$

1. Show that the series

$$\sum_{n=0}^{\infty} \sin\left(\frac{x}{2^n}\right)$$

converges uniformly on $[-1, 1]$ to a differentiable function $f(x)$ with $f'(0) = 2$.

2. Let

$$\alpha(x) = \begin{cases} 0 & 0 \le x < 1 \\ x^2 & 1 < x \le 2 \end{cases}.$$

Compute

$$\int_0^2 x \, d\alpha(x).$$

3. Give an example of a subset of $\mathbb{R}$ which is not an F_σ. Prove your assertions.

4. Determine the values of p $(0 < p < \infty)$ such that $t^{-2} \sin t \in L^p$. Give proofs.

5. Let $f_n(t) = t^{d + 1/n}$.
   a) Prove that $\{f_n\}$ converges in L^1 as $n \to \infty$, if $d > -1$.

   b) Prove that $\{f_n\}$ converges uniformly on $[0, 1]$ if and only if $d > 0$.

6. Suppose $f(t)$ is a measurable function on $[0, 1]$, $1 < p < \infty$, and $t^d \cdot f(t) \in L^p$, for every $d > 0$. Prove that $f(t) \in L^p$ if and only if there is a constant c such that

$$\int |t^d f(t)|^p \, dt \le c$$

for all $d > 0$.

7. If m is a positive measure and $E_1, E_2, \ldots$ are measurable sets satisfying

$$\sum_{j=1}^{\infty} m(E_j) < \infty,$$

prove that

$$\lim_{n\to\infty} m\left(\bigcup_{j=1}^{n} E_j\right) = m\left(\bigcup_{j=1}^{\infty} E_j\right).$$

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8. If $f(t) \in L^1$, show that

$$\int_0^1 f(t)\,(t - s)^{-1/2}\,dt \in L^1$$

for almost all $s \in [0, 1]$.

9. Suppose M is a closed subspace of L^∞ such that every $f \in M$ is continuous on some open interval I_f containing $1/2$. Prove that there is an interval $(1/2 - \epsilon, 1/2 + \epsilon)$ on which every $f \in M$ is continuous.

10. Define a map $T: L^2 \to L^2$ by $Tf = \varphi f$, where $\varphi \in L^\infty$. Prove that T maps L^2 onto L^2 if and only if $1/\varphi \in L^\infty$.