1. a) Let \( f(x) \) be a bounded, piecewise continuous function on \([0, 1]\). Prove
\[
\int_0^1 f(x) \, dx \leq \int_0^1 e^{f(x)} \, dx.
\]

b) Let \( \{a_1, \ldots, a_n\} \) be a set of positive numbers. Prove
\[
\frac{1}{a_1} + \cdots + \frac{1}{a_n} \leq \sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.
\]

2. For \( Y \) a set and \( \mathcal{A} \subseteq \mathcal{P}(Y) \) (\( \mathcal{P}(Y) \) = set of subsets of \( Y \)) let \( \mathcal{B}(\mathcal{A}) \) be the sigma algebra generated by \( \mathcal{A} \). For \( i = 1, 2, \) let \( X_i \) be a set and \( \mathcal{B}_i \) a sigma algebra on \( X_i \). Suppose \( \mathcal{A} \subseteq \mathcal{B}_2 \) and \( \mathcal{B}(\mathcal{A}) = \mathcal{B}_2 \). Prove the following: A function \( f : X_1 \rightarrow X_2 \) is measurable if and only if \( f^{-1}(\mathcal{A}) \subseteq \mathcal{B}_1 \).

3. Suppose \( f(x) \) is continuous and \( f \in L^1(\mathbb{R}) \).

Prove \( \lim_{\varepsilon \to 0} \frac{2}{\varepsilon} \int_0^\infty e^{-\varepsilon u} \cos xu \left( \int_0^\infty f(t) \cos tu \, dt \right) \, du = f(x) \).

4. For \( \{f_n : n \geq 1\} \) a sequence in \( L^2(X, \mu) \) and \( f \in L^2(X, \mu) \), prove the following: The sequence converges \( \{f_n : n \geq 1\} \) converges to \( f \) in mean if and only if a) \( \lim_{n \to \infty} (f_n, g) = (f, g) \) for any \( g \in L^2(X, \mu) \) and b) \( \lim_{n \to \infty} ||f_n||_2 = ||f||_2 \).

5. Let \( \{f_n : n \geq 1\} \) be a sequence of continuous functions on the open interval \((a, b)\) with \( a < b \). Suppose \( \sup_n f_n(c) < \infty \) for any \( a < c < b \). Prove there exist \( \alpha \) and \( \beta \) with \( a < \alpha < \beta < b \) such that
\[
\sup_{\alpha < x < \beta} \left( \sup_n f_n(x) \right) < \infty.
\]
6. Consider the transformation $T : (x, y, z) \rightarrow (u, v, w)$ where $u = x + y + z$, $v = x^2 + y^2 + z^2$ and $w = x^3 + y^3 + z^3$.
   a) Prove $T$ maps some neighborhood of $(-1, 0, 1)$ one to one onto a neighborhood of $(0, 2, 0)$.
   b) Does $T$ map a neighborhood of $(0, 0, 2)$ one to one onto a neighborhood of $(2, 4, 8)$?

7. Let $S$ be a subspace of $L^2([0, 1])$, and suppose this is a constant $K$ such that $|f(x)| \leq K\|f\|_2$ for all $f \in S$ and almost all $x \in [0, 1]$. Prove $\dim S \leq K^2$.

8. a) Let $q(x)$ and $p(x)$ be continuous functions on $[a, b]$ with $p(x)$ positive and monotonically decreasing. Prove
   
   $$\int_a^b p(x) q(x) \, dx = p(a) \int_a^c q(x) \, dx$$
   
   for some $a \leq c \leq b$.

   b) Prove $\sqrt[\infty]{\int_0^1 \sin x \, dx} \lim_{A \to \infty} \int_a^A \frac{\sin x}{x} \, dx$ exists.

9. Suppose $f$ is Lebesgue integrable on $[a, b]$ and $F$ is defined by
   $$F(x) = \int_a^x f(t) \, dt.$$  
   Prove $F$ is continuous and of bounded variation on $[a, b]$.

10. If $F(x, y) = \int_x^{x^2 + y^2} e^{-yt^2 - xt} \, dt$ evaluate $F_x(x, y)$ and $F_y(x, y)$. 