Real

Real Analysis Preliminary Examination

September 1991

Instructions: Do all of the problems 1 - 4 and do any four of the problems 5 - 9.

1. Let $f(0,0) = 0$ and $f(x, y) = \frac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$.

   a) Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(x, y) = (0, 0)$.

   b) Is f differentiable at $(0,0)$?

2. Let $F: \mathbb{R}^5 \longrightarrow \mathbb{R}^2$ be given by

$$F(x_1, x_2, y_1, y_2, y_3) = (2e^{x_1} y_1 - y_3^2,\ x_2 \cos x_1 - 5x_1 + y_2^2 - y_3).$$

   a) Show that there is a function

$$g: \mathcal{N} \longrightarrow \mathbb{R}^2,$$

   where $\mathcal{N} \subset \mathbb{R}^3$ is a neighborhood of $(2, -1, 2)$, such that

$$g(2, -1, 2) = (0, 1)$$

   and

$$F(g(\bar{y}), \bar{y}) = 0 \text{ for all } \bar{y} \in \mathcal{N}.$$

   b) Compute $g'(2, -1, 2)$.

3. a) Show that the sequence

$$f_n(x) = \frac{x}{1 + nx^2},\ n = 1, 2, 3, \ldots$$

   converges uniformly to a function $f(x)$ on $\mathbb{R}$.

   b) Is it true that $\lim_{n\to\infty} f_n'(x) = f'(x)$ for all $x \in \mathbb{R}$?

Real

4. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}$. Show that the set

$$A = \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

is a measurable set.

5. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $f: X \longrightarrow \mathbb{R}$ a non-negative measurable function. Let

$$A = \{(x, t) \in X \times \mathbb{R} \mid 0 \leq t \leq f(x)\}.$$

Prove A is measurable with respect to the product measurer $\mu \times \lambda$ on $X \times \mathbb{R}$ (λ is Lebesgue measure on $\mathbb{R}$), and that

$$\mu \times \lambda(A) = \int_X f \, d\mu.$$

6. Let $\mathbb{N}$ denote the natural numbers $\{0, 1, 2, \ldots\}$ and $\mathcal{F}$ the σ-algebra of all subsets of $\mathbb{N}$. For a non-negative sequence $b = \{b_0, b_1, b_2, \ldots\}$, define a measure on $\mathcal{F}$ by

$$\mu_b(E) = \sum_{n \in E} b_n \quad \text{for } E \in \mathcal{F}.$$

If $c = \{c_0, c_1, c_2, \ldots\}$ is another non-negative sequence, characterize when μ_c is absolutely continuous with respect to μ_b and find $d\mu_c/d\mu_b$ in this case.

7. Give an outline of how the Fourier Transform is defined as a map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

8. a) Let H be a complex Hilbert space and $V \subset H$ a closed proper subspace. Show there exists $w \in V^{\perp}$ with $w \neq 0$.

b) Use part a) to prove that if $\ell: H \longrightarrow \mathbb{C}$ is a complex continuous linear functional on H, then there is a vector $v \in H$ such that $\ell(x) = \langle x, v\rangle$ for all $x \in H$.

9. State and prove the closed graph theorem.