Probability Theory, Ph.D Qualifying, Spring 2024

Completely solve any five problems.

1. If $\{X_n\}$ are iid random variables with $P(X_1 = 0) < 1$ and $S_n = X_1 + X_2 + \cdots + X_n$, then for every $c > 0$, there exists an integer $n = n_c$ such that $P(|S_n| > c) > 0$.

2. (a) Given a random variable X with finite mean square. Let $\mathcal{D}$ be a σ-algebra. Show that $E[X|\mathcal{D}]$ is the minimizer of $E(X - \xi)^2$ over all $\mathcal{D}$-measurable r.v.s ξ, i.e.,

$$E(X - E[X|\mathcal{D}])^2 \leq E(X - \xi)^2$$

for all $\mathcal{D}$-measurable r.v.s ξ.

(b) Let $(\Omega, \mathcal{F}, P)$ denote a probability space. Suppose $f : R^n \times \Omega \to R$ is a bounded $\mathcal{B}(R^n) \times \mathcal{C}$ measurable function and X be a n-dimensional $\mathcal{D}$ measurable random variable. Assume $\mathcal{C}$ and $\mathcal{D}$ are independent. If $g(x) := Ef(x, \omega)$, then

$$g(X) = E[f(X, \omega)|\mathcal{D}], \text{ a.s.}$$

3. Show that random variables $X_n,\ n \geq 1$, and X satisfy $X_n \to X$ in distribution iff

$$E[F(X_n)] \to E[F(X)]$$

for every continuous distribution function F.

4. Let $\{X_n\}$ be a sequence of iid random variables with $E|X_1| = \infty$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Show that

$$P\left(\limsup_n \frac{|S_n|}{n} = \infty\right) = 1.$$

5. Let $\{X_n\}$ be a sequence of iid random variables with $EX_1 = 0$. Prove that (a) the sequence $\{\frac{S_n}{n}\}$ is uniformly integrable; (b) $\frac{E|S_n|}{n} \to 0$.

6. Let $\{X_n\}$ be iid r.v.s with distribution $F(x)$ having finite mean μ and variance $\sigma^2 > 0$. Let $S_n = X_1 + \cdots + X_n$. Show that

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \text{ in distribution as } n \to \infty.$$

Here $N(0,1)$ is a standard normal random variable.

7. Let $X_1, X_2, \ldots$ be a sequence of independent r.v.s with $EX_i = 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ and $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. Show that $\phi(S_n)$ is an $\mathcal{F}_n$–submartingale for any convex ϕ provided that $E|\phi(S_n)| < \infty$ for all n.