Topology Qualifying Exam

August, 2013

Justify all the calculations and state the theorems you use in your answers.

1. Prove that the product of two compact spaces is compact.

2. Give an example of a topological space that is connected but not path connected. Make sure you prove that your example satisfies the property.

3. Prove that the Euler characteristic of a compact surface with boundary which has k boundary components is $\leq 2-k$.

4. Give the definition of a covering space $\hat{X}$ (and covering map $p : \hat{X} \to X$) for a topological space X. Prove that if X is simply connected, and $(\hat{X}, p)$ is a covering space for X, then p is a homeomorphism. Make your proof as elementary as possible.

5. Use covering spaces to show that any free group is a subgroup of F_2, the free group on 2 generators.

6. Let X be a space obtained by attaching two 2-cells to the torus $S^1 \times S^1$, one along a simple closed curve $1 \times S^1$ and the other along $S^1 \times 1$. Calculate the fundamental group and homology groups of X.

7. Use Mayer-Vietoris sequence to calculate the homology groups of S^n.

8. Compute the homology groups $H_i(\mathbb{R}P_n/\mathbb{R}P_m;\mathbb{Z})$ of the quotient space $\mathbb{R}P_n/\mathbb{R}P_m$, for $m < n$. Use cellular homology, using the standard CW structure on $\mathbb{R}P_n$ with $\mathbb{R}P_m$ as its m-skeleton.

9. Let S be an oriented surface and let $f : S \to S$ be a continuous map homotopic to the identity that has no fixed points. Find all the possible values for the genus of S. Prove your answer is true.