Instructions: Work as many as you can out of 9 problems, counted equally.

1. a) Let \( p \) and \( q \) be primes with \( p < q \). Let \( G \) be a group of order \( pq \). Prove that if \( p \) does not divide \( q - 1 \), then \( G \) is abelian.
b) Sketch a construction of a non-abelian group of order 39. Give a presentation for this group.

2. a) Let \( G \) be a group with a subgroup \( H \) of finite index, say \( n \). Let \( N = \bigcap_{x \in G} xHx^{-1} \). Prove that \( N \) is a normal subgroup of \( G \), and that \( G/N \) is isomorphic to a subgroup of the symmetric group \( S_n \).
b) Prove that there is no simple group of order 48.

3. Let \( M \) be a finitely generated module over a P.I.D. \( R \), and let \( M_t \) denote the submodule of torsion elements in \( M \). Prove that \( M \) is the direct sum of \( M_t \) and a free module. (You may assume the theorem that a finitely generated torsion-free module over a P.I.D. is free.)

4. Let \( GF(q) \) denote the finite field with \( q \) elements. Let \( f(x) = x^q - x \).
a) Factor \( f(x) \) into a product of irreducible polynomials over \( GF(3) \).
b) Which of the roots of \( f(x) \) generate the multiplicative group of non-zero elements of \( GF(9) \), considered as a splitting field of \( f(x) \) over \( GF(3) \).
c) What is the Galois group of \( f(x) \) over \( GF(3) \)?

5. Let \( A = A(a, b) \) be the matrix \( \begin{pmatrix} a + b & b & b \\ a - b & a & a \\ a + b & b + 1 & b \end{pmatrix} \), where \( a \) and \( b \) are elements of a field \( F \).
a) What are the possibilities for the rank of \( A \)?
b) Let \( F = GF(9) \). Let \( V \) be the subset of \( F^3 \) consisting of pairs \((a, b)\) such that the matrix \( A(a, b) \) has less than maximal rank. Describe \( V \). How many elements does it have?

6. Let \( F_n \) denote a cyclotomic extension of the rationals of order \( n \) (i.e. a splitting field of \( x^n - 1 \) over the rationals).
a) Determine the Galois group of \( F_8 \) over the rationals, and find all intermediate fields.
b) Do the same for \( F_7 \). If \( \zeta \) is a primitive 7th root of unity, determine the minimal polynomial over the rationals of \( \zeta + \zeta^{-1} \).

7. Let \( A \) be the matrix \( \begin{pmatrix} 1 & 0 & 0 & -3 \\ -1 & 0 & 1 & 0 \\ -1 & 2 & 2 & -3 \\ -1 & 0 & 0 & 1 \end{pmatrix} \), and let \( B \) be the matrix \( \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -8 & 0 & 4 \end{pmatrix} \). Find the elementary divisors, invariant factors, characteristic polynomials, and minimal polynomials for each of \( A \) and \( B \). Are \( A \) and \( B \) similar? Why or why not?

8. Give precise statements of the following results.
a) The classification of finitely generated modules over a P.I.D. (One classification system is sufficient.)
b) The Cayley-Hamilton theorem.
c) The spectral theorem (finite-dimensional, real or complex, case).

9. a) Define the dimension of a (finite-dimensional) real vector space \( V \), and then indicate (with brief justification) the dimensions of the real vector spaces \( \mathbb{R}^m \oplus \mathbb{R}^n \) (direct sum) and \( \mathbb{R}^m \otimes \mathbb{R}^n \) (tensor product).
b) Define what it means for a module \( M \) over a commutative ring \( R \) to be noetherian, and then give an equivalent condition.
c) Define an elementary Jordan matrix (a single Jordan block) over a field \( F \).