Algebra Qualifying Examination
January 2006

Instructions: Work as many as you can out of 9 problems, counted equally.

1. a) Let p and q be primes with $p < q$. Let G be a group of order pq. Prove that if p does not divide $q-1$, then G is abelian.
b) Sketch a construction of a non-abelian group of order 39. Give a presentation for this group.

2. a) Let G be a group with a subgroup H of finite index, say n. Let $N = \bigcap_{x \in G} xHx^{-1}$. Prove that N is a normal subgroup of G, and that G/N is isomorphic to a subgroup of the symmetric group S_n.
b) Prove that there is no simple group of order 48.

3. Let M be a finitely generated module over a P.I.D. R, and let M_t denote the submodule of torsion elements in M. Prove that M is the direct sum of M_t and a free module. (You may assume the theorem that a finitely generated torsion-free module over a P.I.D. is free.)

4. Let $GF(q)$ denote the finite field with q elements. Let $f(x) = x^9 - x$.
a) Factor $f(x)$ into a product of irreducible polynomials over $GF(3)$.
b) Which of the roots of $f(x)$ generate the multiplicative group of non-zero elements of $GF(9)$, considered as a splitting field of $f(x)$ over $GF(3)$.
c) What is the Galois group of $f(x)$ over $GF(3)$?

5. Let $A = A(a,b)$ be the matrix $\begin{pmatrix} a+b & b & b \\ a-b & a & a \\ a+b & b+1 & b \end{pmatrix}$, where a and b are elements of a field F.
a) What are the possibilities for the rank of A?
b) Let $F = GF(9)$. Let V be the subset of F^2 consisting of pairs (a,b) such that the matrix $A(a,b)$ has less than maximal rank. Describe V. How many elements does it have?

6. Let F_n denote a cylotomic extension of the rationals of order n (i.e. a splitting field of $x^n - 1$ over the rationals).
a) Determine the Galois group of F_8 over the rationals, and find all intermediate fields.
b) Do the same for F_7. If ζ is a primitive 7th root of unity, determine the minimal polynomial over the rationals of $\zeta + \zeta^{-1}$.

7. Let A be the matrix $\begin{pmatrix} 1 & 0 & 0 & -3 \\ -1 & 0 & 1 & 0 \\ -1 & 2 & 2 & -3 \\ -1 & 0 & 0 & 1 \end{pmatrix}$, and let B be the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -8 & 0 & 4 \end{pmatrix}$. Find the elementary divisors, invariant factors, characteristic polynomials, and minimal polynomials for each of A and B. Are A and B similar? Why or why not?

8. Give precise statements of the following results.
a) The classification of finitely generated modules over a P.I.D. (One classification system is sufficient.)
b) The Cayley-Hamilton theorem.
c) The spectral theorem (finite-dimensional, real or complex, case).

9. a) Define the <u>dimension</u> of a (finite-dimensional) real vector space V, and then indicate (with brief justification) the dimensions of the real vector spaces $\mathbb{R}^m \oplus \mathbb{R}^n$ (direct sum) and $\mathbb{R}^m \otimes \mathbb{R}^n$ (tensor product).
b) Define what it means for a module M over a commutative ring R to be <u>noetherian</u>, and then give an equivalent condition.
c) Define an <u>elementary Jordan matrix</u> (a single Jordan block) over a field F.