Qualifying Examination Problems for Numerical Analysis

Spring, 2022

Numerical Linear Systems

[1] Let A be an invertible matrix and $\tilde{A}$ be a perturbation of A satisfying $\|A^{-1}\|\,\|A-\tilde{A}\| < 1$. Suppose that x and $\tilde{x}$ are the exact solutions of $Ax = b$ and $\tilde{A}\tilde{x} = \tilde{b}$, respectively. Show that

$$\frac{\|x-\tilde{x}\|}{\|x\|} \le \frac{\text{cond}(A)}{1-\ \text{cond}(A)\frac{\|A-\tilde{A}\|}{\|A\|}} \left[\frac{\|A-\tilde{A}\|}{\|A\|} + \frac{\|b-\tilde{b}\|}{\|b\|}\right].$$

[2] Define the singular value decomposition (SVD) for a rectangular matrix A of size $m \times n$. (3 points) Explain how to use SVD for solving a linear least squares problem (7 points)

[3] Explain how to use the power method to find the leading eigenvalue of a symmetric matrix A. (3 points) Furthermore, explain how to use the power method to find all other eigenvalues of A. (7 points)

Polynomial Interpolation and Approximation

[4] (10 points) Prove that for any polynomial q of degree $\le n-1$

$$\sum_{i=0}^{n} q(x_i) \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)^{-1} = 0,$$

where $x_0, x_1, \cdots, x_n$ are distinct.

[5] (10 points) Let $x_0, x_1, \cdots, x_n$ be given points in $[a,b]$ and f be a continuous function over $[a,b]$. What is the Lagrange interpolatory polynomial p_n of degree n satisfying

$$p_n(x_i) = f(x_i), i = 0, \cdots, n?(3points)$$

Suppose that $f \in C^{n+1}(a,b)$. What is the error $f(x) - p_n(x)$? (3 points). Furthermore, using $p_n'(x)$ to approximate $f'(x)$ based on $(x_i, f_i), i = 0, \cdots, n$, what is the error term $p_n'(x) - f'(x)$ if f is $(n+1)$ times differentiable? (4 points)

Numerical Integration

[6] (10 points) Define Gauss Quadrature formula for $\displaystyle\int_{-1}^{1} f(x)dx$, i.e.

$$\sum_{i=0}^{n} c_i f(x_i) \approx \int_{-1}^{1} f(x)dx$$

using polynomials of degree $n \geq 1$. (5 points). Show that the coefficients c_i are nonnegative. (5 points)

Spline Approximation

[7] Find the smoothness conditions to ensure that a spline function $s(x)$ defined on $[a, b] \cup [b, c]$ by

$$s(x) = \begin{cases} \sum_{i=0}^{n} c_i B_{i,n-i}(x, a, b) & \text{if } x \in [a, b) \\ \sum_{i=0}^{n} d_i B_{i,n-i}(x, b, c), & \text{if } x \in [b, c] \end{cases}$$

is $C^1[a, c]$, where $B_{i,n-i}(x, a, b) = \binom{n}{i}(x - a)^i (b - x)^{n-i}/(b - a)^n, i = 0, \cdots, n$ are Bernstein polynomials of degree n and similar for $B_{i,n-i}x, b, c)$.

Numerical Solution of ODE

[8] (10 points) Explain a method to solve a second order ODE (IVP) numerically:

$$\begin{cases} \dfrac{d^2y}{d^2x} = f(x, y, y'), x \in [a, b] \\ y(a) = \alpha \text{ and } y'(a) = \beta. \end{cases} \tag{1}$$

Be sure to explain its convergence of your method.