Numerical Analysis Qualifying Exam

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Please attempt all problems. Each problem is worth 10 points.

[1] Suppose that a square matrix A is strictly diagonally dominant. Show that when applying Gaussian elimination procedure with partial pivoting to A, there is no partial pivoting needed.

[2] Let A be an $n \times n$ non-singular matrix, and consider iterative methods of the form

$$Mx^{n+1} = b + Nx^n$$

where $A = M - N$.

(a) Assuming M is non-singular, state a sufficient condition that ensures convergence of the iterates to the solution of $Ax = b$ for any starting vector x^0.

(b) Describe the matrices M and N for (i) Jacobi iteration and (ii) Gauss-Seidel iteration.

[3] Let A be a real $m \times n$ matrix with singular value decomposition(SVD) $A = U\Sigma V^T$. Denote the nonzero diagonal entries of Σ by $\sigma_1, \cdots, \sigma_r$. Let

$$A^+ = V\Sigma^+ U^\top$$

be the pseudo inverse of A, where $\Sigma^+ = \text{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots, \frac{1}{\sigma_r}, 0, \cdots, 0)$ of size $m \times n$. Show that

$$AA^+A = A \quad \text{and} \quad (A^+A)^\top = A^+A.$$

[4] Solve the following:

(a) Find and solve the normal equations used to determine the coefficients for a straight line that fits the following data in the least squares sense.

x_i	$f(x_i)$
-1	2
0	3
1	3
2	4

(b) Let A be an $m \times n$ matrix, with $m > n$, and the columns of A being linearly independent. Given the QR factorization of A, where the columns of Q are orthonormal and R is upper triangular, what equations must you solve to find the least squares solution of the over-determined system of equations $Ax = b$?

[5] Use Steepest Descent Method to solve

$$g(\mathbf{x}^*) = \min_{\mathbf{x} \in \mathbf{R}^n} g(\mathbf{x})$$

where $g(\mathbf{x}) = \frac{1}{2}\mathbf{x}^t A\mathbf{x} - \mathbf{x}^t\mathbf{b}$, $n = 3$, and

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Start with $\mathbf{x}^0 = (0, 0, 0)^t$ and do three iterations.

[6] Let x and y be two vectors in $\mathbf{R}^2$. Suppose that $\|x\|_2 = \|y\|_2$. Show that there exists a unitary matrix H such that $Hx = y$.

[7] Let $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ be a partition of $[a, b]$. For $f \in C^1[a, b]$, let S_f be the C^1 cubic interpolatory spline of f, i.e.,

$$S_f(x_i) = f(x_i), \quad S_f'(x_i) = f'(x_i), \quad i = 0, 1, \cdots, n+1$$

and $S_f(x)|_{[x_i, x_{i+1}]}$ is a cubic polynomial, $i = 0, \cdots, n$. Suppose that $f \in C^2[a, b]$. Show that

$$\int_a^b \left| \frac{d^2}{dx^2}(f(x) - S_f(x)) \right|^2 dx \leq \int_a^b \left| \frac{d^2}{dx^2} f(x) \right|^2 dx.$$

[8] Consider the forward and backward difference operators D^+ and D^- defined by

$$D^+ f(x) = \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad D^- f(x) = \frac{f(x) - f(x-h)}{h}.$$

(a) Assuming f is smooth, derive asymptotic error expansions for each of these operators.

(b) What combination of $D^+f(x)$ and $D^-f(x)$ gives a second order accurate approximation to the derivative $f'(x)$? Justify your answer.

[9] Consider the integration formula

$$\int_{-1}^{1} f(x)dx \approx f(\alpha_1)\beta + f(\alpha_2)\beta.$$

(a) Determine α_1, α_2, and β so that this formula is exact for all quadratic polynomials.

(b) What is the expected order of a composite integration method based upon the formula with coefficients derived in (a)?

[10] Consider the ordinary differential equation

$$y'(t) = f(t, y(t)),\ \ y(t_0) = y_0.$$

(a) Give a derivation of a multi-step method (Adams-Bashforth) of order 2 to solve this problem.

(b) Find the leading term of the local truncation error. What is the global error of the method?