1. Let \( \chi_{[0,\infty)} \) denote the characteristic function of \([0,\infty)\). Show that there is no everywhere continuous function \( f \) on \( \mathbb{R} \) such that \( f(x) = \chi_{[0,\infty)}(x) \) for almost every \( x \in \mathbb{R} \) (with respect to Lebesgue measure).

2. Let \( \{E_n\}_{n \in \mathbb{N}} \) be a countable family of Lebesgue measurable subsets of \( \mathbb{R}^d \) with
\[
\sum_{n=1}^{\infty} m(E_n) < \infty
\]
where \( m \) denotes Lebesgue measure on \( \mathbb{R}^d \) and let
\( E = \{ x \in \mathbb{R}^d : x \in E_n \text{ for infinitely many } n \in \mathbb{N} \} \).
(a) Show that \( E = \bigcap_{n=1}^{\infty} \bigcup_{n \geq N} E_n \) and deduce that \( E \) is Lebesgue measurable with \( m(E) = 0 \).
(b) Show that
\[
\chi_E(x) = \limsup_{n \to \infty} \chi_{E_n}(x)
\]
for all \( x \in \mathbb{R}^d \) where, for any subset \( A \) of \( \mathbb{R}^d \), \( \chi_A \) denote the characteristic function of \( A \).

3. Prove that if \( g \) is continuous with compact support on \( \mathbb{R}^d \), then
\[
\lim_{n \to \infty} \int |g(n^{1/n}x) - g(x)| \, dx = 0
\]
and deduce from this that if \( f \in L^1(\mathbb{R}^d) \), then
\[
\lim_{n \to \infty} \int |f(n^{1/n}x) - f(x)| \, dx = 0.
\]

4. Let \( f \) be the function defined over \( \mathbb{R} \) by
\[
f(x) = \begin{cases} 
x^{-1/2} & \text{if } 0 < x < 1, \\
0 & \text{otherwise}. 
\end{cases}
\]
For a given enumeration \( \{q_n\}_{n=1}^{\infty} \) of the rationals \( \mathbb{Q} \), let
\[
F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x + q_n).
\]
(a) Prove that \( F \) is a Lebesgue integrable function on \( \mathbb{R} \) and hence that the series defining \( F \) converges for almost every \( x \in \mathbb{R} \).
(b) Show, however, that this series is unbounded on every open interval, and in fact, any function \( G \) that agrees with \( F \) almost everywhere must be unbounded on every open interval.

5. Let \( \{u_j\}_{j=1}^{\infty} \) be an orthonormal basis for \( L^2(\mathbb{R}^d) \). Prove that the collection \( \{u_{j,k}\}_{j,k=1}^{\infty} \) with
\[
u_{j,k}(x,y) := u_j(x)u_k(y)
\]
forms an orthonormal basis for \( L^2(\mathbb{R}^d \times \mathbb{R}^d) \).

6. Let \( (X, \mathcal{B}, \mu) \) be a measure space with \( \mu(X) = 1 \). Prove that for any integrable function \( f : X \to \mathbb{C} \)
\[
\mu\left( \{ x \in X : |f(x)| \geq \frac{1}{2} \|f\|_1 \} \right) \geq \max \left\{ \frac{\|f\|_1}{2\|f\|_\infty}, \frac{\|f\|^2_1}{4\|f\|^2_2} \right\}.
\]