# Algebra Prelim

Fall 2004

1. How many $p$-Sylow subgroups does $A_5$ have, where
   (a) $p = 2$,
   (b) $p = 3$,
   (c) $p = 5$?

2. Let $P$ be a Sylow subgroup of a finite group $G$. Prove that $N(N(P)) = N(P)$, where $N(\cdot)$ denotes the normalizer.

3. Let $R$ be a commutative ring, and let $a \in R$ be an element such that for all positive integers $n$, $a^n \neq 0$. Prove that $a$ lies in a prime ideal.

4. Give an example of each of the following phenomena:
   (a) an imbedding of commutative rings, $R \subset S$, and an element $p \in R$ such that $p$ is prime in $R$ but not in $S$,
   (b) an imbedding of commutative rings, $R \subset S$, and an element $p \in R$ such that $p$ is prime in $S$ but not in $R$.

5. (a) Show that $\mathbb{Q}[\sqrt{2} + \sqrt{3} + \sqrt{5}] = \mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}]$.
   (b) Compute the Galois group of $\mathbb{Q}[\sqrt{2} + \sqrt{3} + \sqrt{5}]/\mathbb{Q}$.
   (c) Find all intermediate fields $K$, that is $\mathbb{Q} \subset K \subset \mathbb{Q}[\sqrt{2} + \sqrt{3} + \sqrt{5}]$.

6. State the structure theorem for finitely generated modules over a principal ideal domain. Find the decomposition of the $\mathbb{Z}$-module presented by the following generators and relations.

$$\begin{aligned} 4w + 3x + 6z &= 0 \\ 3w + 12y + 3x + 6z &= 0 \\ 6y &= 0 \\ -3w - 3x + 6y &= 0 \end{aligned}$$

7. Let $A$ be a nilpotent matrix (i.e., $A^n = 0$ for some $n > 0$), with real entries, and let $p(x) = \sum a_i x^i$ be a polynomial with $a_0 \neq 0$. Prove that the matrix $p(A)$ is invertible.

8. Let $E$ be a field, let $H$ be a finite subgroup of $Aut(E)$, and let $Fix(H)$ denote the fixed field. Prove that $E$ is algebraic over $Fix(H)$.

9. Let

$$A = \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ q & 0 & 1 \end{pmatrix}$$

where $p$ and $q$ are real numbers. For each of the following conditions, determine, with proof, the values of $p$ and $q$ for which the condition holds.

(a) The rank of $A$ is two.

(b) $A$ is invertible.

(c) $A$ is diagonalizable.