Algebra Preliminary Exam

Monday, September 16, 1996

Problems 1 and 2 are worth 20 points; the others are 12 points each.

1. State the following theorems, defining the terms in brackets:
   (a) The Sylow Theorem (“three parts”) [Sylow p-subgroup]
   (b) The Spectral Theorem for a self-adjoint operator on a finite-dimensional vector space over $\mathbb{C}$; also give its interpretation for Hermitian matrices [Hermitian inner product, self-adjoint operator, Hermitian matrix, unitary matrix]
   (c) The Fundamental Theorem of Galois Theory [separable, normal, and galois extensions of fields; galois group]
   (d) The Structure Theorem for finite abelian groups, and for finitely generated modules over a PID; explain why the former is a corollary of the latter. [free module, torsion module]
   (e) The Fundamental Theorem on Symmetric Polynomials. [elementary symmetric function]

2. Quick examples: Justify your answers briefly.
   (a) Evaluate the following determinants:
      \[
      \begin{vmatrix}
      1 & 2 & 3 & 4 \\
      2 & 3 & 4 & 5 \\
      3 & 4 & 5 & 6 \\
      4 & 5 & 6 & 7 \\
      \end{vmatrix}
      \quad
      \begin{vmatrix}
      1 & 2 & 2^2 & 2^3 \\
      1 & 3 & 3^2 & 3^3 \\
      1 & 4 & 4^2 & 4^3 \\
      1 & 5 & 5^2 & 5^3 \\
      \end{vmatrix}
      \]
   (b) List the 5 isomorphism types of groups of order 8; for each, give a property (or properties) which distinguishes it from the others
   (c) Find the eigenvalues and eigenvectors of the following matrix, and determine its Jordan Canonical form:
      \[
      \begin{pmatrix}
      12 & 25 \\
      -4 & -8 \\
      \end{pmatrix}
      \]
   (d) Let $\zeta$ be a primitive 25th root of unity. Determine the degree of the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$, and describe its Galois group.
   (e) Let $G$ be a group with a (right) action on a group $N$. Define the semidirect product $N \rtimes G$.

3. Let $V$ and $W$ be finite-dimensional vector spaces over a field $K$; let $V^*$ be the dual space of $V$, and $\text{Hom}(V,W)$ be the space of $K$-vector space homomorphisms from $V$ to $W$. Show that $V^* \otimes_K W$ is canonically isomorphic to $\text{Hom}(V,W)$. 

4. Prove that every finite group of prime-power order is solvable.

5. Let $R$ be a commutative ring with unit. Show that for any $t \in R$ which is not nilpotent, there is a prime ideal of $R$ which does not contain any power of $t$. Use this to show that the intersection of all prime ideals of $R$ is precisely the set of nilpotent elements of $R$ [which is called the nilradical of $R$].

6. Let $R$ be a Noetherian ring; prove that the polynomial ring $R[x]$ is also Noetherian.

7. (a) Explain what it means for a polynomial to be solvable by radicals.

(b) Consider $f(x) = x^7 - 16x + 10$: find its Galois group (over $\mathbb{Q}$), and determine whether or not $f(x)$ is solvable by radicals.