# Complex Analysis Qualifying Exam — Spring 2022

*Show work and carefully justify/prove your assertions. For example, if you use a theorem that has a name, mention the name. Arrange your solutions in numerical order even if you do not solve them in that order.*

1. (10 points) Use complex analysis to compute $I = \displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx$, where $a$ and $b$ are positive constants. Hint: Start by considering $\dfrac{\cos ax - \cos bx}{x^2}$ as the real part of an appropriate complex valued function.

2 (10 points) Let $f(z) = \sum_{n=0}^\infty c_n z^n$ be analytic and one-to-one in $|z| < 1$ with real part $u(z)$ and imaginary part $v(z)$. For $0 < r < 1$, let $D_r$ be the disc $|z| < r$. Prove that the area $A_r$ of $f(D_r)$ is finite and is given by the following formula:

$$\iint_{f(D_r)} dudv = \pi \sum_{n=1}^\infty n|c_n|^2 r^{2n}.$$

3. (10 points) Let $a_n(z)$ be a sequence of analytic functions on an open set $\Omega$ such that $\displaystyle\sum_{n=0}^\infty |a_n(z)|$ converges uniformly on its compact subsets. Show that $\displaystyle\sum_{n=0}^\infty |a_n'(z)|$ also converges uniformly on compact subsets of $\Omega$.

4. (10 points) Show that if $f(z)$ and $g(z)$ are holomorphic functions on an open and connected set $\Omega$ such that $f(z)g(z) = 0$, then either $f(z)$ or $g(z)$ is identically zero.

5. (10 points) Give the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)}$ in each of the following two annuli

$$\text{(i) } \{z : 0 < |z| < 1\}, \ \text{(ii) } \{z : 1 < |z|\}$$

6. (10 points) Find a conformal map from the intersection $D$ of $|z - i| < 2$ and $|z + i| < 2$ to the upper half plane.

7. (10 points) Show that $z = 0$ is an essential singularity for the function $f(z) = e^{\frac{1}{\sin z}}$.