Complex Analysis Qualifying Examination

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All problems are of equal weight. Please arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.

1. (a) Prove that if $|w_1| = c|w_2|$ where $c > 0$, then $|w_1 - c^2 w_2| = c|w_1 - w_2|$.

   (b) Prove that if $c > 0$, $c \neq 1$ and $z_1 \neq z_2$, then $\left|\dfrac{z - z_1}{z - z_2}\right| = c$ represents a circle. Find its center and radius.

2. Compute the following integral carefully justifying each step:
$$\int_0^\infty \frac{\log x}{1 + x^3}.$$

3. (a) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,
$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

   (b) Deduce Liouville's theorem from (a).

4. Suppose that f is holomorphic in an open set containing the closed unit disc, except for a simple pole at $z = 1$. Let $f(z) = \sum_{n=1}^{\infty} c_n z^n$ denote the power series in the open unit disc. Show that $\lim_{n\to\infty} c_n = -\lim_{z\to 1}(z - 1)f(z)$.

5. Find a conformal map that maps the region $\{z \mid \mathrm{Re}(z) > 0, |z - 1/2| > 1/2\}$ to the upper half plane.

6. Prove the *open mapping theorem for holomorphic functions*: If f is a non-constant holomorphic function on an open set U in $\mathbb{C}$, then $f(U)$ is also an open set.

7. Let f be analytic on a bounded domain D, and assume also that f that is continuous and nowhere zero on the closure $\overline{D}$. Show that if $|f(z)| = M$ (a constant) for z on the boundary of D, then $f(z) = e^{i\theta}M$ for z in D, where θ is a real constant.