Complex Analysis Qualifying Exam — Spring 2021

All problems are of equal weight. Please arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.

1. Let z_1 and z_2 be two complex numbers.
   (a) Show that $|z_1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|)$.
   (b) Show that if $|z_1| < 1$ and $|z_2| < 1$, then $\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$.
   (c) Assume that $z_1 \neq z_2$. Show that $\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = 1$ if only if $|z_1| = 1$ or $|z_2| = 1$.

2. Evaluate the integral $\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\cosh(x)} dx$ where $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and ξ is real.
   Hint: Use an appropriate rectangular contour containing $[-R, R]$ as one side.

3. Suppose $f(z)$ is entire and there exist $A, R > 0$ and natural number N such that
$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$
   Show that (a) f is a polynomial and (b) the degree of f is at least N.

4. Let $f(z) = u + iv$ be an entire function such that $u(x, y) = Re(f(x + iy))$ is a polynomial in x, y. Show that $f(z)$ is a polynomial in z.

5. Let $f(z)$ be a holomorphic map of the open unit disk $\mathbb{D}$ into itself. Show that for any two points z and w in $\mathbb{D}$,
$$\left|\frac{f(w) - f(z)}{1 - \overline{f(w)} f(z)}\right| \leq \left|\frac{w - z}{1 - \bar{w} z}\right|$$
   and the inequality is strict for $z \neq w$ except when f is linear fractional transformation mapping the unit disk into itself.

6. Suppose $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on the unit disk $\mathbb{D}$, and $f(z)$ is a holomorphic function on the unit disk $\mathbb{D}$. Show that the following are equivalent.
   (a) $\{f_n(z)\}$ converges to $f(z)$ uniformly on compact subsets in $\mathbb{D}$.
   (b) $\int_{|z|=r} |f_n(z) - f(z)|\,|dz|$ converges to 0 if $0 < r < 1$.

7. Let R be the intersection of the right half plane and the outside of the circle $|z - \frac{1}{2}| = \frac{1}{2}$ with the line segment [1.2] removed, i.e.
$$R = \left\{z : \Re ez > 0 \text{ and } |z - \frac{1}{2}| > \frac{1}{2}\right\} \setminus \{z = x + iy : \ 1 \leq x \leq 2 \text{ and } y = 0\}.$$
   Find a conformal map from R to the upper half plane.