Qualifying Examination in Algebra, January 2008

1. Let G be a group. Let $H \triangleleft G$ and $K \triangleleft G$ be distinct normal subgroups such that G/H and G/K are simple. Prove that $G/K \simeq H/(H \cap K)$.

2. State the decomposition theorem for finitely generated modules over a principal ideal domain, and use it to classify abelian groups of order 72.

3. Find the Galois group of $x^4 - 2$ over $\mathbb{Q}$. That is, find generators and relations for the Galois group, and state how those generators act on a set of generators of the splitting field.

4. Let R be a domain. Prove that the following are equivalent:
a) For every positive integer n, every submodule $M \subset R^n$ is free.
b) R is a principal ideal domain.
(Hint: Consider the exact sequence $0 \to R^{n-1} \to R^n \to R \to 0$.)

5. Let G be a group of order $3p^k$, where p is a prime number and $k \geq 1$. Prove that G is not simple. Be sure to consider all cases.

6. Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree 4. Assume that for all $n \in \mathbb{Z}$, $f(n) \neq 0$. Assume also that $f(0)$ is even, and $f'(0)$ and $f(1)$ are odd. Show that f is irreducible in $\mathbb{Q}[x]$. (Hint: Consider the reduction of f modulo 2.)

7. Let k be an algebraically closed field. Let $G = GL_n(k)$ be the group of invertible $n \times n$ matrices over k, let Y be the set of all $n \times n$ matrices over k, and let $X \subset Y$ be the set of nilpotent $n \times n$ matrices.
i) Give the definition of a group acting on a set.
ii) Show that G acting on Y by conjugation satisfies the definition of group action, and that X is G-stable.
iii) Show that X has finitely many G-orbits.
iv) Show that the two matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

belong to the same orbit.

8. Let F be a field of characteristic zero.
i) Let N_i, $i = 1, 2$, be finite Galois extensions of F. Explain why it makes sense to speak of the field generated by N_1 and N_2, even if N_1 and N_2 are defined only up to isomorphism. Show furthermore that the field generated by N_1 and N_2 is Galois.
ii) For $f \in F[x]$, let E_f denote the splitting field of f over F. Let f and g be polynomials over F, such that $E_f \cap E_g = F$. Prove that $F[\alpha + \beta] = F[\alpha, \beta]$, where α, β are roots of f and g respectively.