1. Let \( f \in \mathbb{Q}[x] \) be an irreducible polynomial, and let \( L \) be a finite Galois extension of \( \mathbb{Q} \). Let \( f(x) = g_1(x)g_2(x) \cdots g_r(x) \) be a factorization of \( f \) into irreducibles in \( L[x] \).
   (a) Prove that each of the factors \( g_i(x) \) has the same degree.
   (b) Give an example to show that if \( L \) is not Galois over \( \mathbb{Q} \), the conclusion of part (a) need not hold.

2. Let \( G \) be a group of order 96.
   (a) Show that \( G \) has either one or three 2-Sylow subgroups.
   (b) Show that either \( G \) has a normal subgroup of order 32 or a normal subgroup of order 16.

3. Consider the polynomial \( f(x) = x^4 - 7 \) in \( \mathbb{Q}[x] \), and let \( E/\mathbb{Q} \) be the splitting field of \( f \).
   (a) What is the structure of the Galois group of \( E/\mathbb{Q} \)?
   (b) Give an explicit description of all of the intermediate subfields \( \mathbb{Q} \subset K \subset E \) in the form \( K = \mathbb{Q}(\alpha), \mathbb{Q}(\alpha, \beta), \ldots \), where \( \alpha, \beta \), etc. are complex numbers. Describe the corresponding subgroups of the Galois group.

4. Let \( F \) be a field and \( T \) and \( n \times n \) matrix with entries in \( F \). Let \( I \) be the ideal consisting of all polynomials \( f \in F[x] \) such that \( f(T) = 0 \). Show that the following statements are equivalent about a polynomial \( g \in I \):
   (a) \( g \) is irreducible,
   (b) if \( k \in F[x] \) is nonzero and of degree strictly less than \( g \), \( k(T) \) is an invertible matrix.

5. Let \( T \) be a \( 5 \times 5 \) complex matrix with characteristic polynomial \( \chi(x) = (x - 3)^5 \), and minimal polynomial \( m(x) = (x - 3)^2 \). Determine all possible Jordan forms of \( T \).

6. Let \( G \) be a group, and let \( H, K < G \) be subgroups of finite index. Show that \( [G : H \cap K] \leq [G : H][G : K] \).

7. Give a careful proof that \( \mathbb{C}[x, y] \) is not a principal ideal domain.

8. Let \( R \) be a commutative ring without unit, such that \( R \) does not contain a proper maximal ideal, and \( R \) is not the zero ring. Prove that for all \( x \in R \), the ideal \( xR \) is proper. You may assume the axiom of choice.