(1) (a) (5 points) State the structure theorem for finitely generated modules over a principal ideal domain.
(b) (5 points) Find the decomposition of the \( \mathbb{Z} \)-module \( M \) generated by \( w, x, y, \) and \( z \), and satisfying the relations
\[
3w + 12y + 3x + 6z = 0 \\
6y = 0 \\
-3w - 3x + 6y = 0.
\]

(2) (10 points) Let \( R \) be a commutative ring and let \( M \) be an \( R \)-module. Recall that for \( \mu \in M \) the annihilator of \( \mu \) is the set \( \text{Ann}(\mu) = \{ r \in R : r\mu = 0 \} \). Suppose that \( I \) is an ideal in \( R \) which is maximal with respect to the property that there exists a nonzero element \( \mu \in M \), such that \( I = \text{Ann}(\mu) \). Prove that \( I \) is a prime ideal in \( R \).

(3) (a) (5 points) Give the definition that a group \( G \) must satisfy to be solvable.
(b) (10 points) Show that every group \( G \) of order 36 is solvable. \( \text{Hint: You may assume that } S_4 \text{ is solvable.} \)

(4) (15 points) Consider the matrix
\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]
(a) Find the Jordan Normal Form of \( A \) regarded as a matrix over \( \mathbb{C} \), the complex numbers.
(b) Find the Jordan Normal Form of \( A \) regarded as a matrix over \( \mathbb{F}_5 \), the field with five elements.

(5) (15 points) Let \( F \subset L \) be fields such that \( L/F \) is a Galois field extension with Galois group equal to \( D_8 = \langle \sigma, \tau : \sigma^4 = \tau^2 = 1, \sigma \tau = \tau \sigma^3 \rangle \). Show that there are fields \( F \subset E \subset K \subset L \) such that \( E/F \) and \( K/E \) are Galois extensions, but \( K/F \) is not Galois.

(6) (15 Points) Let \( C/F \) be an algebraic field extension. Prove that the following are equivalent:
(a) Every nonconstant polynomial \( f \in F[x] \) factors into linear factors in \( C[x] \).
(b) For every (not necessarily finite) algebraic extension \( E/F \) there is a ring homomorphism \( \alpha : E \to C \) that is the identity on \( F \). \( \text{Hint: Use Zorn’s lemma.} \)

(7) (10 Points) Let \( R \) be a commutative ring.
(a) Say what it means for \( R \) to be a unique factorization domain (UFD);
(b) Say what it means for \( R \) to be a principal ideal domain (PID);
(c) Give an example of a UFD that is not a PID. Prove that it is not a PID.

(8) (10 Points) Let \( p \) and \( q \) be distinct primes. Let \( k \) denote the smallest positive integer such that \( p \) divides \( q^k - 1 \). Prove that no group of order \( pq^k \) is simple.