1. List all groups of order 14 up to isomorphism. Carefully justify your answer.

2. Please show that $S_4$ is a solvable non-abelian group.

3. Please identify all the conjugacy classes of elements in the group $S_5$. Provide an explicit representative for each class, and justify that your list is complete.

4. Determine the group of units in each of the following rings:
   
   $\mathbb{Z}[i]$, $\mathbb{F}_3[x]/(x^2 + 1)$, $\mathbb{F}_5[x]/(x^2 + 1)$, $\mathbb{F}_{81}$, $\mathbb{Z}/81\mathbb{Z}$.

5. Let $R$ be any principal ideal domain. Let $n > 0$ and $A \in M_n(R)$ denote a square $(n \times n)$-matrix with coefficients in $R$. Consider the $R$-module $M := R^n/\text{Im}(A)$.
   
   (a) Give a necessary and sufficient condition for $M$ to be a torsion module (i.e., every non-zero element of $M$ is torsion). Justify your answer.
   
   (b) Let $F$ be a field, and let now $R := F[x]$, the ring of polynomials in one variable with coefficients in $F$. Given an example of an integer $n > 0$ and a $(n \times n)$-square matrix $A \in M_n(R)$ such that $M := R^n/\text{Im}(A)$ is isomorphic as $R$-module to $R \times F$.

6. Let $R$ and $S$ be two commutative rings (with multiplicative identity).
   
   (a) Prove that when $R$ is a field, every non-zero ring homomorphism $\phi : R \to S$ is injective.
   
   (b) Does (a) still hold if we only assume that $R$ is a domain? If yes, prove it, and if not, provide a counter-example.

7. Let $K$ be a field. State the main theorem of Galois theory for a finite field extension $L/K$.
   
   (b) Let $\zeta_{43} := \exp(2\pi i/43)$. Describe the group of all field automorphisms $\sigma : \mathbb{Q}(\zeta_{43}) \to \mathbb{Q}(\zeta_{43})$.
   
   (c) How many proper subfields are there in the field $\mathbb{Q}(\zeta_{43})$?

8. Suppose that $\alpha$ is a root in $\mathbb{C}$ of $P(x) := x^{17} - 2$. How many field homomorphisms are there from $\mathbb{Q}(\alpha)$ to
   
   (a) $\mathbb{C}$,
   
   (b) $\mathbb{R}$,
   
   (c) $\overline{\mathbb{Q}}$, an algebraic closure of $\mathbb{Q}$?
   
   (Justify your answers.)

9. Let $V \neq (0)$ be a finite dimensional vector space over an algebraically closed field $k$. Please prove that every linear map $L : V \to V$ must have an eigenvector $v \in V$ (please provide a simple proof, without using the Cayley-Hamilton theorem, for instance.)
   
   Does this statement remain true if $k$ is not algebraically closed? If yes, prove it, and if not, provide a counter-example.

10. Let $M \in M_5(\mathbb{R})$ be a square $(5 \times 5)$-matrix with real coefficients, defining a linear map $L : \mathbb{R}^5 \to \mathbb{R}^5$. Assume that when considered as an element of $M_5(\mathbb{C})$, then the scalars 0, 1 + $i$, and 1 + 2$i$, are eigenvalues of $M$.
    
    (a) Show that the associated linear map $L$ is neither injective nor surjective.
    
    (b) Compute the characteristic polynomial and the minimal polynomial of $M$.
    
    (c) How many fixed points can $L$ have (that is, how many solutions to the equation $L(v) = v$ with $v \in \mathbb{R}^5$)? (Justify.)