1. Let $R$ be a commutative ring and $M$ an $R$-module, $M \neq \{0\}$.
   i) Say what it means for $M$ to have a basis.
   ii) Prove that if $R$ is a field, then $M$ has a (not necessarily finite) basis. Indicate where the hypothesis that $R$ is a field is used.

2. Prove Gauss's lemma: The product of primitive polynomials in $\mathbb{Z}[x]$ is primitive.
   (A polynomial is said to be primitive if the greatest common divisor of its coefficients is 1.)

3. Let $H$ and $K$ be subgroups of a group $G$, such that $K$ is normal in $G$.
   i) Prove that $HK$ is a subgroup of $G$.
   ii) Prove that $HK/K \cong H/(H \cap K)$.

4. Let $E/F$ be a Galois field extension, and let $K/F$ be an intermediate field of $E/F$. Prove that $K$ is normal over $F$ if and only if $Gal(E/K)$ is a normal subgroup of $Gal(E/F)$.

5. Let $A$ be an $n \times n$ matrix over $\mathbb{C}$, such that $A^*A = AA^*$. Prove that $A$ is diagonalizable.

6. Classify all groups of order 55.

7. Let $M = \mathbb{R}[x]/(x - 2)(x + 1) \oplus \mathbb{R}[x]/(x - 2)(x^2 + 3)$. Let $T : M \to M$ denote the $\mathbb{R}$-linear transformation “multiplication by $x$.” Find the following for $T$:
   i) minimal polynomial
   ii) characteristic polynomial
   iii) determinant
   iv) rational canonical form.

8. Let $\zeta_{11} = e^{2\pi i/11}$ (so $\zeta_{11}$ is a primitive 11th root of unity).
   i) Prove that $\mathbb{Q}(\zeta_{11})$ is a Galois extension of $\mathbb{Q}$ and describe the Galois group of this extension.
   ii) Find all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta_{11})$ and write each in the form $\mathbb{Q}(\alpha)$ for some $\alpha$. Prove your answers.