2015 Winter Algebra Qual

1. For a prime p, let G be a finite p-group, and let N be a normal subgroup of G of order p. Prove that N is contained in the center of G.

2. Let $\mathbb{F}$ be a finite field.
   a. Give, with proof, the decomposition of the additive group $(\mathbb{F}, +)$ into a direct sum of cyclic groups.
   b. The exponent of a finite group is the least common multiple of the orders of its elements. Prove that a finite abelian group has an element of order equal to its exponent.
   c. Prove that the multiplicative group $(\mathbb{F}^{\times}, \cdot)$ is cyclic.

3. Let F be a field, let V be a finite dimensional F-vector space, and let $A, B\colon V \to V$ be commuting F-linear maps. Suppose there is a basis $\mathcal{B}_1$ with respect to which A is diagonalizable and a basis $\mathcal{B}_2$ with respect to which B is diagonalizable. Prove that there is a basis $\mathcal{B}_3$ with respect to which A and B are both diagonalizable.

4. Let N be a positive integer, and let G be a finite group of order N.
   a. Let $\operatorname{Sym} G$ be the set of all bijections from G to G, viewed as a group under composition. Note that $\operatorname{Sym} G \cong S_N$. Prove that the Cayley map $C : G \to \operatorname{Sym} G$ given by $g \mapsto (x \mapsto gx)$ is an injective homomorphism.
   b. Let $\Phi : \operatorname{Sym} G \to S_N$ be an isomorphism. For $a \in G$, define $\epsilon(a) \in \{\pm 1\}$ to be the sign of the permutation $\Phi(C(a))$. Suppose that a has order d. Prove that $\epsilon(a) = -1$ if and only if d is even and N/d is odd.
   c. Suppose $N > 2$ and $N \equiv 2 \pmod 4$. Prove that G is *not* simple. (Hint: Use part b).)

5. Let $f(x) = x^4 - 5 \in \mathbb{Q}[x]$.
   a. Compute the Galois group of f over $\mathbb{Q}$.
   b. Compute the Galois group of f over $\mathbb{Q}(\sqrt{5})$.

6. Let F be a field, and let n a positive integer. Consider

$$A = \begin{bmatrix} 1 & \dots & 1 \\ & \ddots & \\ 1 & \dots & 1 \end{bmatrix} \in M_n(F).$$

   Show that A has a Jordan normal form over F, and find it.
   (Suggestion: treat the cases $n \cdot 1 \neq 0$ in F and $n \cdot 1 = 0$ in F separately.)

7. Let R be a commutative ring. Let S be a subset of R which is nonempty, does not contain 0, and for all $x, y \in S$ we have $xy \in S$. Let $\mathcal{I}$ be the set of all ideals I of R such that $I \cap S = \varnothing$. Show that for every ideal $I \in \mathcal{I}$, there is an ideal $J \in \mathcal{I}$ such that $I \subset J$ and J is not properly contained in any other ideal in $\mathcal{I}$. Prove that every such ideal J is prime.

8. Let R be a principal ideal domain, and let M be a finitely generated R-module.
   a. Prove that there are R-submodules $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ of M such that for all $0 \leq i \leq n-1$, M_{i+1}/M_i is cyclic (i.e., generated by a single element).
   b. Is the integer n of part a) uniquely determined by M? (Prove your answer.)