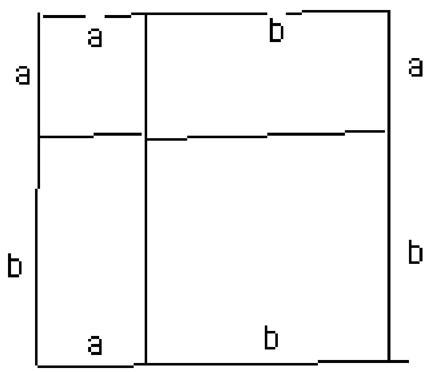
Law of Cosines

Pythagoras says the square on the side oppoiite a right angle "equals" the two squares on the sides containing the angle. If the angle is acute, the square on the side opposite it is smaller than the two squares on the sides containing it, and if obtuse the square it is greater. The law of cosines tells exactly how much less or how much greater, in particular the discrepancy is twice the area of a certain rectangle. These are propositions 12-13, Book II of Euclid. First he proves propositions that amount to the equation (a+b)^2 = a^2 +2ab + b^2, as we will also do.

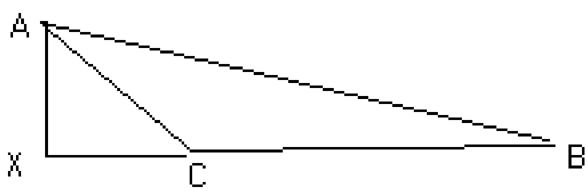
Proposition: If a segment be divided at any point into lengths a and b, the square on the whole segment equals the squares on the two portions of the segment plus twice the rectangle with sides a and b.
I.e. (a+b)^2 = a^2 +2ab + b^2.
proof: Look at this picture.



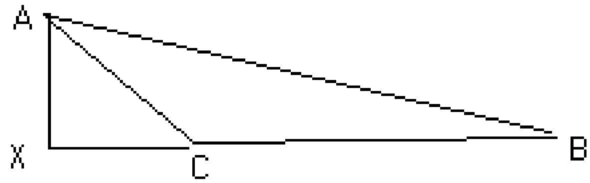
QED.

Remark: I was a senior in college when I first saw this picture, in a class with Professor Jerome Bruner, famous psychologist of learning.

Theorem: If an angle C in a triangle ABC is obtuse, the square on the side opposite C equals the sum of the squares on the two adjacent sides, plus twice the rectangle whose sides are one of the adjacent sides and the projection onto it of the other adjacent side.
proof:



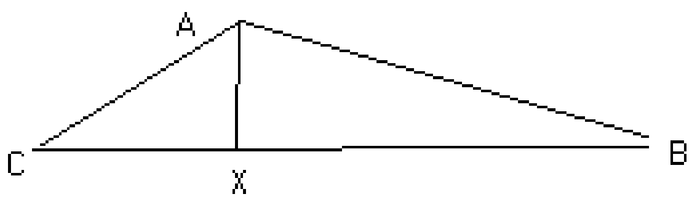
I.e. in the picture below,



the square with side AB is equal to the square with side AC plus the square with side BC plus twice the rectangle with sides BC and CX.

By Pythagoras the square on AB equals the sum of the squares on AX and BX. From the proposition, this equals the square on AX plus that on CX plus that on BC plus twice the rectangle with sides CX and BC. BY Pythagoras again, this equals the square on AC + that on BC + twice the rectangle with sides CX and BX. **QED.**

Theorem: If an angle C in a triangle ABC is acute, the square on the side opposite C plus twice the rectangle whose sides are one of adjacent sides and the projection onto it of the other adjacent side, equals the sum of the squares on the two adjacent sides.
proof:



I.e. in the picture above, the square on AB plus twice the rectangle with sides CX and BC, equals the square on AC plus the square on AB.

By the proposition and Pythagoras, the square on AC + the square on BC equals the square on AX + twice the square on CX + the square on BX + twice the rectangle with sides CX and BX. By Pythagoras this equals the square on AB + twice (the square on CX + the rectangle with sides CX and BX) which equals the square on AB + twice (the rectangle with sides CX and CX+BX) = the square on AB + twice the rectangle with sides CX and BC. **QED.**

Transitivity of “congruent dissections”.

Since all our arguments about “equal” figures, depend on transitivity of the relation we gave for “equal” figures, we will prove that property.

Theorem: If two figures are both “equal” to a third, in the sense of having dissections into triangles congruent to a common set of triangles, then the two original figures are also equal to each other.

Proof: Let A,B, and C be figures (sums of triangles) and assume that C is both a (non overlapping) sum of the triangles T1,....,Tn, and also a (non overlapping) sum of the triangles S1,....,Sm. Assume further that A is a non overlapping sum of triangles T1’,...,Tn’ congruent to T1,..,Tn, and that B is a sum of triangles S1’’,....,Sm’’ congruent to S1,....,Sm, i.e. that A is equal to C and also B is equal to C.

Then C is also a sum of the figures Rij, where Rij = (Ti intersected with Sj), and we claim that both A and B are also sums of figures congruent to the figures Rij, for i=1,...,n, and j =

1,...,m.

This is easy, since A is the sum of the triangles Ti’, and since each of the triangles Ti is the sum of the figures Rij, for j = 1,...,m, so also is Ti’ the sum of figures R’ij congruent to the Rij. Thus A is the sum of all these R’ij. Similarly B is the sum of triangles S’’j congruent to the Sj, and each S’’j is the sum of figures R’’ij congruent to the Rij, for i=1,...,n. Thus B is the sum of the R’’ij, congruent to Rij, and hence also to R’ij.

Finally the intersection Rij of two triangles, is convex, hence “easily shown” to be a sum of triangles. So at last A and B are (differently) sums of triangles congruent to a common set of triangles. **QED.**