Complex analysis qualifying examination, Fall 2009

Do all problems, and justify your assertions.

Problem 1. Let $f$ be analytic on the unit disk. Is it possible that $f\left(\frac{1}{n}\right)$ takes the following values for $n = 2, 3, 4, \ldots$? Why or why not?

a) $(-1)^n$

b) $e^{-n}$ for $n$ even, and 0 for $n$ odd

c) $\frac{1}{\lfloor \sqrt{n} \rfloor}$ (here $\lfloor x \rfloor$ is the greatest integer not greater than $x$)

d) $\frac{n-2}{n-1}$

Problem 2. Use complex analysis to prove the Fundamental Theorem of Algebra: if $p(z)$ is a non-constant polynomial with complex coefficients, then there exists a complex number $z_0$ such that $p(z_0) = 0$.

Problem 3. Compute the integral

$$\int_0^\infty \frac{\cos x}{(1 + x^2)^2} \, dx$$

Problem 4. Give the Laurent series for the following functions, and characterize their singularities at zero (essential, removable or pole):

a) $\frac{\sin^2 z}{z}$

b) $ze^{-\frac{1}{z^2}}$

c) $\frac{1}{z(4-z)}$

Problem 5. Let $D$ be the domain in the complex plane consisting of the interior of the unit disk, with the subdisk of radius $\frac{1}{2}$ and center $\frac{1}{2}$ removed. Construct an analytic function that maps $D$ conformally onto the first quadrant $\Re z > 0, \Im z > 0$. (See figure.)

Problem 6. Give Laplace’s equation in polar coordinates. In other words, if $x = r \cos \theta, y = r \sin \theta$, write down a relation equivalent to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

in terms $r, \theta$, and the partial derivatives of $f$ with respect to $r, \theta$.  

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