The Prime Number Theorem and Its History

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Directed Reading Program, 2017
With Kübra Benli

1. What are prime numbers?
2. How many primes are there?
3. The Prime Number Theorem

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3. The Prime Number Theorem

Definition

An integer $p > 1$ is called a prime number in case there is no divisor d of p satisfying $1 < d < p$.

A prime number only has two positive factors: 1 and itself.

Example

2, 3, 5, 7, 11, 13, 17, 19, 23, 29,...,$2^{74207281}$-1 (22,338,618 digits),...

Prime numbers are important because they are building blocks for the integers:

Theorem (Fundamental Theorem of Arithmetic)

Every integer $n > 1$ can be expressed as a product of primes, and this factorization is unique apart from the order of the prime factors.

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Is there a largest prime?

-Euclid proved the following theorem.

Theorem

There are infinitely many primes.

Proof.

Suppose there are n primes and name them $p_1, p_2, p_3, ..., p_n$. Then let $M = p_1 p_2 ... p_n + 1$. Since M is not divisible by $p_1, p_2, ..., p_n$, M should have a prime factor different from these listed primes. So we obtain a new prime number other than the given ones. Therefore, there are infinitely many primes. □

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How infinite primes are?

Call the number of primes less than or equal to a positive number x, $\pi(x)$, that is, $\pi(x) = \#\{p\text{: prime}|\ p \leq x\}$ for a positive real number x. Euclid's proof can be interpreted into

$$\lim_{x\to\infty} \pi(x) = \infty.$$

But how large $\pi(x)$ is when x is large? For that we seek a fuction $f(x)$ such that

$$\lim_{x\to\infty} \frac{\pi(x)}{f(x)} = 1,$$

in that case we use the notation

$$\pi(x) \sim f(x).$$

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Legendre's First Conjecture

In 1798, Legendre published the first conjecture on the size of $\pi(x)$ in his book *Essai sur la Théorie des Nombres*. Legendre stated the following:

$$\pi(x) \sim \frac{x}{\log x - 1.08366}$$

x	$\pi(x)$	Legendre	%Error
10^3	168	172	2.381
10^4	1229	1231	0.162
10^5	9592	9588	0.042
10^6	78498	78534	0.046
10^7	664579	665138	0.084
10^8	5761455	5769341	0.137
10^9	50847534	50917519	0.138
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Gauss's Li(x)

Gauss was also studying prime tables and came up with a different estimate for $\pi(x)$ (perhaps first considered in 1791), communicated in a letter to a friend in 1849 and first published in 1863. Gauss conjectured the following:

$$\pi(x) \sim \int_2^x \frac{1}{\log t} dt \sim \frac{x}{\log x}$$

The integral in the middle is called the logarithmic integral and denoted by $\mathrm{Li}(x)$.

x	$\pi(x)$	$\mathrm{Li}(x)$	%Error
10^3	168	178	5.9523
10^4	1229	1246	1.3832
10^5	9592	9630	0.3961
10^6	78498	78628	0.1656
10^7	664579	664918	0.0510
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Chebyshev's Approximation

Chebyshev made the first real progress toward a proof of the prime number theorem in 1850. He showed there exist positive constants $a \leq 1 \leq b$ such that

$$a\frac{x}{\log x} < \pi(x) < b\frac{x}{\log x}.$$

He also showed that **IF** $\frac{\pi(x)}{\frac{x}{log(x)}}$ had a limit, then its value must be one.

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For a positive integer n, von Mangoldt function $\Lambda(n)$ is defined as the following:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^a \text{ for some } a \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

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For a positive integer x, $\psi(x)$ is defined as the following:

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

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The Prime Number Theorem

In 1896, Hadamard and independently de la Vallée Poussin completely proved the Prime Number Theorem using ideas introduced by Riemann's $\zeta(s)$ function. We now have

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1$$

In other words,

$$\pi(x) \sim \frac{x}{\log x}$$

Summary

Theorem (Infinitely Many Primes)

$$\lim_{x\to\infty} \pi(x) = \infty.$$

Theorem (Prime Number Theorem)

$$\pi(x) \sim \frac{x}{\log x}$$

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