Topology Qualification Exam, Spring 2023

Instructions: You can assume homology groups and fundamental groups of a point and wedges of spheres in all dimensions. Everything else should be computed. All problems have equal weight.

1. Prove that a metric space is Hausdorff.

2. Let \( \{X_i \mid i \in I\} \) be a collection of topological spaces indexed by an indexing set \( I \). Let \( X = \prod_{i \in I} X_i \) be the Cartesian product. Recall that there are two natural topologies one might put on \( X \), the box topology, with basis equal to the set of sets of the form \( \prod_{i \in I} U_i \) for all possible open \( U_i \subset X_i \), and the product topology, with the same basis elements except that in each product \( \prod_{i \in I} U_i \), all but finitely many \( U_i \) are required to equal the total space \( X_i \). Give an example where \( X \) with the product topology is not homeomorphic to \( X \) with the box topology.

3. Describe a path-connected 3-sheeted covering space \( p : \tilde{X} \to X \) of \( X = \mathbb{R}P^2 \vee S^1 \). Make sure to describe both the space \( \tilde{X} \) and the map \( p \). Let \( x_0 \in X \) denote the point at which the wedge operation is performed to create \( \mathbb{R}P^2 \vee S^1 \). Fix some \( \tilde{x}_0 \in p^{-1}(x_0) \) and explicitly describe the subgroup \( p_\ast(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0) \) in terms of the description of \( \pi_1(X, x_0) \) as \( (\mathbb{Z}/2\mathbb{Z}) \ast \mathbb{Z} \).

4. Explicitly describe a path-connected space \( X \) with basepoint \( x_0 \in X \) such that \( \pi_1(X, x_0) \cong \mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z}) \).

5. Consider a regular octagon \( P \) in the plane with opposite sides identified by a rigid translation of the plane. In other words, consider the equivalence relation \( \sim \) on \( P \) where for two distinct points \( p, q \in P \), \( p \sim q \) if and only if \( p \) and \( q \) are on the boundary of \( P \) and there is a rigid translation of the plane taking one edge of \( P \) to an opposite edge and taking \( p \) to \( q \). This produces an orientable surface \( \Sigma = P/\sim \).

   (a) Calculate the genus of \( \Sigma \).

   (b) Let \( \rho : P \to P \) be rotation by \( \pi \) about the center point of \( P \). Note that since \( p \sim q \) implies \( \rho(p) \sim \rho(q) \), \( \rho \) descends to a map \( \rho : \Sigma \to \Sigma \). (You do not need to prove that fact.) How many fixed points does \( \rho : \Sigma \to \Sigma \) have?

   (c) We claim that \( \Sigma/\rho \) is a surface (do not prove this); what is the genus of \( \Sigma/\rho \)?

6. Decompose \( S^1 \times S^n \) as \( (S^1 \times S^n) \cup (S^1 \times S^n) \), where

\[
S^n_\pm = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1 \text{ and } \pm x_0 > -1/2\}.
\]

Use the Mayer-Vietoris sequence for this decomposition to show that \( H_k(S^1 \times S^n) \cong H_{k-1}(S^1 \times S^{n-1}) \) for all \( k \geq 3 \) and for all \( n \geq 1 \). (This is also true for other values of \( k \) and \( n \) but this is the easiest case to prove.)

7. Let \( B \) be the closed unit ball in \( \mathbb{R}^3 \), let \( S \) be the circle of radius \( 1/2 \) centered at the origin in the \( xy \) plane in \( \mathbb{R}^3 \), and let \( P = (0,0,0) \). Compute the homology of \( X = B \setminus (S \cup \{P\}) \).
8. Using cylindrical coordinates \((r, \theta, z)\) on \(S^2\), consider the function \(f_n : S^2 \to S^2\) given by 
\[ f_n(r, \theta, z) = (r, n\theta, z) \]
for some \(n \in \mathbb{Z}\). Use cellular homology to compute all homology groups of the space \(X\) obtained by gluing \(B^3\) to \(S^2\) using the map \(f_n\) (thought of as a map from the boundary of \(B^3\) to \(S^2\)).