TOPOLOGY QUALIFYING EXAM: SPRING 2009

I. [10 points] Let (X,d) be a compact metric space, and let $f : X \to X$ be an isometry: for all $x, y \in X$, $d(x,y) = d(f(x), f(y))$. Show that f is a bijection.

II [10 points]. a) Show that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.
b) Give an example which shows that the "Hausdorff" hypothesis in part a) is necessary.

III. [10 points]. Show that a connected, normal topological space with more than a single point is uncountable.

IV [10 points]. Let X be the one-point union of $S^1 \times S^1$ and S^1.
a) Compute the fundamental group of X.
b) Compute the homology groups of X.

V [15 points].
a) What is the degree of the antipodal map on the n-sphere? (no justification required)
b) Define a CW-complex homeomorphic to real projective n-space $\mathbb{RP}^n$.
c) Use parts a) and b) to compute the integral homology groups of $\mathbb{RP}^n$.

VI [15 points]. Let X be a CW-complex and $\pi : Y \to X$ be a covering space.
a) Show that Y is compact iff X is compact and π has finite degree.
b) Assume that π has finite degree d. show that $\chi(Y) = d\chi(X)$.
c) Let $\pi : \mathbb{RP}^N \to X$ be a covering map. Show that if N is even, π is a homeomorphism.

VII [10 points]. How many surfaces are there, up to homeomorphism which are: connected, compact, possibly with boundary, possibly nonorientable and with Euler characteristic -3? Describe one representative from each class.

VIII [20 points] View the torus T as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$. Let A be a 2×2 matrix with $\mathbb{Z}$-coefficients.
a) Show that the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ descends to a continuous map $\mathcal{A} : T \to T$.
(b) Show that, with respect to a suitable basis for $H_1(T, Z)$, the matrix A represents the map induced on H_1 by $\mathcal{A}$.
(c) Find a necessary and sufficient condition on A for $\mathcal{A}$ to be homotopic to the identity.
(d) Assume additionally that $\mathcal{A}$ is a homeomorphism, that $\det A = 1$, and that all entries of A are nonnegative. Find a necessary and sufficient condition on A for $\mathcal{A}$ to be homotopic to a map with no fixed points.