Topology Qualifying Exam
August 12, 2010

1. (a) What does it mean to say a topological space is *compact*?

   (b) Show that a product space $X \times Y$ is compact if and only if both X and Y are compact.

2. (a) What does it mean to say a topological space is *connected*?

   (b) What does it mean to say a topological space is *path-connected*?

   (c) Prove that if X is path-connected, then X is connected.

   (d) Either prove or give a counterexample to the converse of part (c).

3. Fix an integer n. Let X be the surface obtained from the cylinder $S^1 \times [0,1]$ by identifying the point $(z, 0)$ with the point $(z^n, 1)$ for all $z \in S^1$. Find the fundamental group of X.

4. Calculate the homology groups of $S^2 \times S^2$.

5. (a) What is meant by the *degree* of a continuous map $f : S^n \to S^n$?

   (b) Let $f : S^n \to S^n$ be a continuous map with no fixed points. Prove that f has degree $(-1)^{n+1}$.

6. Suppose that X has universal cover $p : \tilde{X} \to X$ and let A be a subspace of X with $p(\tilde{a}) = a \in A$. Show that there is a group isomorphism

$$\ker(\pi_1(A, a) \xrightarrow{i_*} \pi_1(X, a)) \cong \pi_1(p^{-1}(A), \tilde{a})$$

   where $i : A \to X$ is the inclusion map.

7. (a) State the classification theorem for compact connected surfaces (with or without boundary).

   (b) Show that any compact connected surface with a nonempty boundary is homotopy equivalent to a wedge of circles. (Hint: you may assume that any compact connected surface without boundary is given by identifying edges of a polygon in pairs.)

   (c) For each surface in your classification, say how many circles are needed in the wedge from part (b). (Hint: you should be able to do this even if you have not done part (b).)

8. For which compact connected surfaces Σ (with or without boundary) does there exist a continuous map $f : \Sigma \to \Sigma$ that is homotopic to the identity map and has no fixed points? Explain your answer fully.