Topology Qualification Exam, Spring 2024

Instructions: You can assume homology groups and fundamental groups of a *point* and *wedges of spheres in all dimensions*. Everything else should be computed.

1. (a) Show that, if X is a Hausdorff space and if A is a subset of X that is compact with respect to the subspace topology, then A is closed as a subset of X.

   (b) Give an example showing that part (a) would no longer be true if the Hausdorff assumption on A were dropped.

2. Let (X, d) be a metric space and let $\mathcal{B} = \{B_\alpha\}_{\alpha \in A}$ be a collection of nonempty open subsets that is a base for the topology on X. For each α, let $x_\alpha \in B_\alpha$. Prove that $\{x_\alpha\}_{\alpha \in A}$ is a dense subset of X.

3. Let A and B be two Möbius strips, and let X be the space formed by gluing A and B together by a homeomorphism between their boundary circles.

   (a) Compute the fundamental group of X.

   (b) Compute all homology groups of X.

   (c) Identify X in terms of the classification of surfaces.

   (d) Find a connected 2-sheeted covering space for X.

4. Prove that if X is a topological space and A is a subset such that A has more path components than X does, then the relative homology $H_1(X, A)$ is nonzero.

5. Let X be the topological space obtained from $\mathbb{R}^3$ by removing x-, y- and z-axis. Compute the fundamental group of X.

6. (a) Let $\rho_3 : S^1 \to S^1$ be the $2\pi/3$-rotation, and X_3 be the topological space obtained from $[0, 1] \times S^1$ by identifying each $(1, x)$ with $(1, \rho_3(x))$ for all $x \in S^1$. Compute $\pi_1(X_3)$.

   (b) Let Y be the topological space obtained from attaching X_3 to $S^1 \times S^1$ by identifying $\{0\} \times S^1$ with $\{x\} \times S^1$ via the identity map. Compute all homology groups of Y.

7. Determine whether the following statements are true or false. Prove it if it is true, and find a counter example if it is false.

   (a) For $n > 1$, every continuous map from S^n to $T^n = S^1 \times S^1 \times \cdots \times S^1$ is nullhomotopic.

   (b) For $n > 1$, every continuous map from T^n to S^n is nullhomotopic.

8. Compute all homology groups of $S^1 \times \mathbb{R}P^2$ using cellular homology.