Instructions: Do all 5 problems.

1. Suppose $A$ is an uncountable subset of $\mathbb{R}$ and $A'$ is the set of limit points of $A$. Prove $A \cap A'$ is uncountable.

2. If $0 < p \leq 1$, prove that $\frac{\sin x}{x^p}$ is not integrable on $(0, \infty)$, but

$$\int_0^\infty \frac{\sin x}{x^p} \, dx = \lim_{r \to \infty} \int_0^r \frac{\sin x}{x^p} \, dx$$

exists.

3. Suppose $\{f_n : n \geq 1\}$ is a sequence of continuous functions on $\mathbb{R}^m$ with the property that for every $x \in \mathbb{R}^m$,

$$\sum_{n=1}^\infty |f_n(x) - f_{n-1}(x)| \leq \frac{1}{1 + |x|^2}.$$ 

Prove the sequence $\{f_n : n \geq 1\}$ converges to a continuous function.

4. If $\mu$ is Lebesgue measure on $\mathbb{R}$ and $f \in L(\mathbb{R}, d\mu)$, prove that $F$ is absolutely continuous where

$$F(x) = \int_{(-\infty, x]} f \, d\mu.$$ 

5. Let $f$ be continuously differentiable on $\mathbb{R}$, and suppose that $f$ and $f'$ are both integrable. Prove

$$\lim_{k \to \infty} \int_{-\infty}^\infty \sin(kx)f(x) \, dx = 0.$$


Analysis Qualifying Exam: complex analysis part
January, 2005

Instructions: 5 problems, counted 10 points apiece.

#1. State the most general version that you know of Cauchy’s Integral Formula and then sketch a proof of the simplest version of Cauchy’s Integral Formula.

#2. Evaluate the following integral by the method of residues (with justification).

\[ \int_{-\infty}^{\infty} \frac{1 + x^2}{1 + x^4} \, dx \]

#3. Prove that if a complex power series \( \sum_{n=0}^{\infty} a_n z^n \) converges for some \( z_0 \neq 0 \), then it converges absolutely in the open disk \( \{ z \in \mathbb{C} : |z| < |z_0| \} \) and uniformly on any closed subdisk.

#4. State the general theorem on existence of a Laurent series and determine the Laurent series for \( \frac{z}{1 - z^2} \) on the annulus \( 0 < |z - 1| < 1 \) and on the annulus \( 1 < |z| < \infty \).

#5. Construct a conformal mapping from \( H = \{ z \in \mathbb{C} : Im(z) > 0 \} \) in \( \mathbb{C} \) onto \( G = \{ w \in \mathbb{C} : Re(w) > 0 \text{ and } |w| < 1 \} \). Include a definition of conformal mapping, and indicate why your mapping is conformal.