Analysis Qualifying Exam: real analysis part
January, 2005

Instructions: Do all 5 problems.

1. Suppose A is an uncountable subset of $\mathbb{R}$ and A' is the set of limit points of A. Prove $A \cap A'$ is uncountable.

2. If $0 < p \leq 1$, prove that $\frac{sin\, x}{x^p}$ is not integrable on $(0, \infty)$, but

$$\int_0^\infty \frac{sin\, x}{x^p} dx = \lim_{r \to \infty} \int_0^r \frac{sin\, x}{x^p} dx$$

exists.

3. Suppose $\{f_n : n \geq 1\}$ is a sequence of continuous functions on $\mathbb{R}^m$ with the property that for every $x \in \mathbb{R}^m$,

$$\sum_{n=1}^{\infty} |f_n(x) - f_{n-1}(x)| \leq \frac{1}{1 + |x|^2}.$$

Prove the sequence $\{f_n : n \geq 1\}$ converges to a continuous function.

4. If μ is Lebesgue measure on $\mathbb{R}$ and $f \in L(\mathbb{R}, d\mu)$, prove that F is absolutely continuous where

$$F(x) = \int_{(-\infty,\, x]} f \, d\mu.$$

5. Let f be continuously differentiable on $\mathbb{R}$, and suppose that f and f' are both integrable. Prove

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} sin(kx) f(x) dx = 0.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Analysis Qualifying Exam: complex analysis part
January, 2005

Instructions: 5 problems, counted 10 points apiece.

#1. State the most general version that you know of Cauchy's Integral Formula and then sketch a proof of the simplest version of Cauchy's Integral Formula.

#2. Evaluate the following integral by the method of residues (with justification).

$$\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4}\, dx$$

#3. Prove that if a complex power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$, then it converges absolutely in the open disk $\{z \in \mathbb{C} : |z| < |z_0|\}$ and uniformly on any closed subdisk.

#4. State the general theorem on existence of a Laurent series and determine the Laurent series for $\dfrac{z}{1-z^2}$ on the annulus $0 < |z-1| < 1$ and on the annulus $1 < |z| < \infty$.

#5. Construct a conformal mapping from $H = \{z \in \mathbb{C} : Im(z) > 0\}$ in $\mathbb{C}$ onto $G = \{w \in \mathbb{C} : Re(w) > 0 \text{ and } |w| < 1\}$. Include a definition of conformal mapping, and indicate why your mapping is conformal.