Ph.D. Prelim: Probability Theory, January 2003

(Solve at least 5 problems completely.)

1. (a) If $\{A_n,\ n \geq 1\}$ are events satisfying $\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$ and $\lim_{n\to\infty} P(A_n) = 0$, then $P(A_n,\ i.o.) = 0$.

   (b) If the independent random variables $X_1, \ldots, X_n, \ldots$ satisfy the condition

   $$\mathrm{Var}(X_i) \leq c < \infty,\ i = 1, 2, \ldots,$$

   then the SLLN holds.

2. (a) State (without proof) the Levy continuity theorem regarding a sequence of characteristic functions.

   (b) Let $\{X_n\}$ be iid r.v.s with distribution $F(x)$ having finite mean μ and variance σ^2. Let $S_n = X_1 + \cdots + X_n$. Show that

   $$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \text{ in distribution as } n \to \infty.$$

3. Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables such that

   $$P(0 \leq X_{n+1} \leq X_n) = 1,\ n \geq 1.$$

   Prove that the convergence in probability $X_n \to 0$ as $n \to \infty$ implies

   $$P(X_n \to 0 \text{ as } n \to \infty) = 1.$$

4. If F_n and F are distribution functions, $F_n \to F$ in distribution, and F continuous, show that

   $$\sup_x |F_n(x) - F(x)| \to 0.$$

5. Let $\{X_n\}$ denote a sequence of iid random variables. Show that $E \sup_n |X_n/n| < \infty$ if and only if $E(|X_1| \cdot (\log |X_1|)^+) < \infty$.

6. Given iid random variables $\{X_n\}$, let

   $$\overline{X} = \frac{X_1 + \cdots + X_n}{n}.$$

   Prove that for every $1 \leq k \leq n$, the equality

   $$E[X_k | \overline{X}] = \overline{X}$$

   holds with probability 1.