Analysis Prelim Exam - Spring 2001

1. Prove that there exists a real valued function f, defined on a neighborhood of the origin in $\mathbb{R}^2$, such that

$$f(0,0) = 0 \text{ and } xyf(x,y)^2 - 2f(x,y)^4 - f(x,y) = x^2y^2.$$

In problems 2-5, m stands for Lebesgue measure on $\mathbb{R}$.

2. Suppose that $f(t)$ and $tf(t)$ are in $L^1(\mathbb{R}, dm)$. Let

$$g(x) = \int_{-\infty}^{\infty} f(t)\sin(xt)dm(t).$$

Prove that g is differentiable, and find $g'(x)$.

3. Let $f \in L^1([0,1], dm)$, and for all Borel subsets B of $\mathbb{R}$, define $v(B) = m(f^{-1}(B))$. Prove that for all bounded Borel functions Y on $\mathbb{R}$,

$$\int_{[0,1]} Y(f(x))dm(x) = \int_{\mathbb{R}} Y(x)dv(x).$$

4. Let $g(x) = x^2$, and consider the linear map $T : L^1([0,1], dm) \to \mathbb{R}$ defined by $Tf = \int_{[0,1]} g(x)f(x)dm(x)$. Recall that

$$||T|| \equiv sup\{|T\ f| : f \in L^1([0,1], dm), ||f||_1 \leq 1\}.$$

Compute $||T||$. Note Verify your computation - do not just quote the Riesz representation theorem.

5. Let $f(x) = \frac{1}{1+x^2}$ and $g(x) = e^x$. Let $\mathcal{C} \equiv \{f^{-1}(B) : B \text{ Borel} \subset \mathbb{R}\}$. For each $C \in \mathcal{C}$, let $v_{\mathcal{C}}(C) = \int_C g(x)dm(x)$, and $m_{\mathcal{C}}(C) = m(C)$.
   a) Prove $B \in \mathcal{C}$ if and only if B is Borel and $B = -B$.
   b) Verify that $v_{\mathcal{C}}$ is absolutely continuous with respect to $m_{\mathcal{C}}$, on the σ-algebra $\mathcal{C}$.
   c) Verify that g is not $\mathcal{C}$-measurable.
   d) Compute the Radon-Nikodyn derivative of $v_{\mathcal{C}}$ w.r.t. $m_{\mathcal{C}}$.

6. Evaluate $\int_\gamma \frac{dz}{(z^2+4)^2}$ for each of the following γ. Be sure to state clearly which results you are using to evaluate the integral.
   a) $\gamma(t) = -i + \frac{1}{2}e^{it}, 0 \leq t \leq 2\pi$
   b) $\gamma(t) = -i + 4e^{it}, 0 \leq t \leq 2\pi$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

7. a) Find a linear fractional transformation which maps the upper half plane onto the interior of the unit circle.
   b) Explicitly describe the image of the first quadrant of the unit circle under the above transformation.

8. Evaluate $\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4}\, dx$. Explain your work clearly, and justify your evaluation.

9. Find the Laurent series expansion for $f(z) = \frac{z^2}{z^2-9}$ valid on the annulus $2 < |z-1| < 4$.

10. a) Prove or give a counter example to the following. There is a monic polynomial $p(z)$ of degree n and an $R > 0$ such that $|p(z)| < R^n$ whenever $|z| = R$.