DEPARTMEMT OF MATHEMATICS, UNIVERSITY OF GEORGIA
PRELIMINARY EXAMINATIONS– SPRING 2000

ANALYSIS

NO AIDS.
DO ALL QUESTIONS.
QUESTIONS WILL BE WEIGHTED EQUALLY.

1. Let X be a metric space and $\{x_n\}_1^\infty$ a convergent sequence in X with limit x_0. Prove that the set $C = \{x_0, x_1, x_2, \cdots\}$ is compact.
2. Let n be a positive integer and $0 < \theta < \pi$. Prove that
$$\frac{1}{2\pi i}\int_{|z|=2}\frac{z^n}{1-2z\cos\theta+z^2}dz = \frac{\sin(n\theta)}{\sin\theta},$$
where the circle $|z| = 2$ is oriented counterclockwise.
3. Suppose that f is an entire function such that $|f(z)| \leq C|z|^{1/2}$ for $|z|$ sufficiently large. (C is a constant). What can you conclude about the form of f? Give a proof. (**Hint**: Cauchy integral formula).
4. Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{e^{-itx}}{a^2+x^2}dx$ via residues , where $a > 0$. Justify every step.
5. (a) Is the following a Banach space (with repect to a suitable norm)?
$$B = \{f : \mathbb{R} \to \mathbb{R} \text{ s.t. } f \text{ continuous, and } \lim_{|x|\to\infty} f(x) = 0\}.$$
Justify your answer.
   (b) Suppose f is continuous, and such that $\sup|f \cdot g| \leq C \sup|g|$ for all $g \in B$. Prove that $|f| \leq C$.
6. (a) What is the dual of $L^3(\mathbb{R})$? Give a proof.
   (b) Exhibit an element of the dual of ℓ^∞ that is not in ℓ^1.
7. (a) Show that if $f \in L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R})$, then $f \in L^p(\mathbb{R})$ for all $p_1 \leq p \leq p_2$.
   (b) Produce a function f such that $f \in L^p(\mathbb{R})$ **only** when $p = 2$.
8. Suppose that $h \in C^1[0,1]$ and ν is a finite Borel measure on $[0,1]$. Let $G(x) = \nu([0,x])$. Prove the following integration by parts formula:
$$\int_0^1 h(x)d\nu(x) = h(1)G(1) - \int_0^1 h'(x)G(x)dx.$$
(Hint: Fubini's theorem.)

9. Find a measure μ, singular with respect to Lebesgue measure, such that $\mu(I) > 0$ for every non-empty interval I.
10. Prove that there is **no one-to-one** conformal map of the punctured disc $G = \{z \in \mathbb{C} : 0 < |z| < 1\}$ onto the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.