Real and Complex Analysis Preliminary Examination

Spring, 1999

Name____ Student Id. No.________________

Instruction: There are ten problems in total. Please work as many problems as possible. Use a separate sheet of paper to do each problem and show all your work.

[1] State and prove one of the following theorems:

1. Lusin's Theorem
2. Egoroff's Theorem
3. Fubini's Theorem
4. Lebesgue Dominated Convergence Theorem
5. The Riesz Representation Theorem for L_p.

[2] Show that the set of all discontinuity points of a monotone function is countable.

[3] Suppose that $f_n, n = 1, 2, 3, \cdots$ converge to f in measure and $g_n, n = 1, 2, 3, \cdots,$ converge to g in measure. Here, f_n converges to f in measure if, for any $\epsilon > 0$.

$$\lim_{n \to +\infty} \text{measure}(\{x, |f_n(x) - f(x)| > \epsilon\}) = 0.$$

Suppose that both f and g are bounded almost everywhere. Show that $f_n g_n$ converges to fg in measure.

[4] Suppose that f is Lebesgue integrable on $[a, b]$. Show that

$$\lim_{n \to +\infty} \int_a^b f(x) |\cos(nx)| dx = \int_a^b f(x) dx.$$

[5] Show that if $\{f_n\}$ converges to f in $L_2(a, b)$ weakly and $\|f_n\|_2$ converges to $\|f\|_2$, then f_n converges to f strongly in $L_2(a, b)$.

[6] State and prove one of the following theorems:

1. Cauchy's Integral Theorem
2. Morera's Theorem
3. Rouche's Theorem
4. Weierstrass' Theorem
5. Mittag-Leffler's Theorem

[7] (a) Find an analytic function $f(z)$ at $z = 0$ satisfying $f(1/n) = \frac{n}{n+1}$, $n = 1, 2, \cdots,$. Show that such an analytic function is unique.

(b) Can you find an analytic function $f(z)$ at $z = 0$ satisfying $f(\frac{1}{2n-1}) = 0, f(\frac{1}{2n}) = \frac{1}{2n}$ for $n = 1, 2, \cdots,$? Give your function if you can or give your reason if you can't.

[8] Compute the Laurent series of function $f(z) = \frac{1}{(z-1)(z-2)}$ in annulus $\{z : 1 < |z| < 2\}$ and $\{z : 2 < |z| < \infty\}$.

[9] Use the residuel theorem to compute the following integration:

$$I = \int_0^\infty \frac{\ln x}{(1+x)^3} dx.$$

[10] Find explicitly a conformal mapping of half disk $\{z : |z| < 1, \mathrm{Im}(z) > 0\}$ onto the upper half plane.