Algebra Preliminary Exam

Tuesday, March 31, 1998

Do as many problems as you can. Problem 1 is worth 25 point, the others are worth 10 points each. The number of problems done completely will also be taken into account: one correct problem is better than two half-done problems.
$\mathbb{Z}, \mathbb{Q}$, and $\mathbb{C}$ denote the integers, the rational numbers, and the complex numbers, respectively.

1. (a) Explain why there is a natural one-to-one correspondence between maximal ideals of $\mathbb{C}[x]$ and the elements of $\mathbb{C}$.
   (b) Let R be a ring. Prove that if I is an ideal of R and $I \neq R$, then I is contained in a maximal ideal of R.
   (c) Give an example of a commutative ring R, an R-module M, and an exact sequence of R-modules $0 \to A \to B$ such that $0 \to A \otimes M \to B \otimes M$ is not exact.
   (d) Suppose A is a hermitian (self-adjoint) matrix over the complex numbers. Prove that there is a matrix B such that $A = B^2$.
   (e) Identify the $\mathbb{Z}$-module $\mathbb{Z}[(1+\sqrt{5})/2] \,/\, \mathbb{Z}[\sqrt{5}]$ as a standard finitely generated module over the PID $\mathbb{Z}$.

2. Let R be a commutative ring with 1. Let P be a prime ideal of R. Prove that if there are ideals $I_1, I_2, \ldots, I_n$, such that$P = I_1 \cap I_2 \cap \ldots \cap I_n$, then $P = I_j$ for some j.

3. Let p be a prime and let n be a natural number. Let $\mathrm{GF}(p^n)$ denote the field of order p^n. Prove that the group of automorphisms of $\mathrm{GF}(p^n)$ is cyclic of order n.

4. Suppose that G is a group of order 18. Prove that either G is abelian, or G is isomorphic to the dihedral group D_9, or G is generated by three elements a, b, c, such that
   $a^3 = b^3 = c^2 = 1$, $ab = ba$ and $cac^{-1} = a^q b^r, cbc^{-1} = a^s b^t$, where $\begin{bmatrix} q & r \\ s & t \end{bmatrix}$ is in $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ and has order 2. (If you have time at the end of the test: how many non-isomorphic groups of the latter type are there?)

5. Find a set of matrices over the complex numbers such that any matrix (over $\mathbb{C}$) whose characteristic polynomial equals $(x-2)^3$ is similar (conjugate) to one and only one matrix in your set. Prove your answer.

6. (a) Find the order of the group $\mathrm{SL}_2(\mathbb{Z}/7\mathbb{Z})$ and prove your answer.
   (b) How many Sylow 7-subgroups does $\mathrm{SL}_2(\mathbb{Z}/7\mathbb{Z})$ have? Find one explicitly.

7. Let $\alpha \in \mathbb{C}$ be a root of $x^3 + 2x + 2$.
   (a) Prove that $\mathbb{Q}[\alpha]$ is a field.
   (b) Find $(\alpha^2 + 1)^{-1}$ as a polynomial in α.

8. Let p be an odd prime. Let F be splitting field of $x^p - 1$ over $\mathbb{Q}$. Prove that there is a unique field K between $\mathbb{Q}$ and F which is of degree 2 over $\mathbb{Q}$. Describe this field explicitly when $p = 5$.