Algebra Preliminary Exam

Tuesday, March 31, 1998

Do as many problems as you can. Problem 1 is worth 25 points, the others are worth 10 points each. The number of problems done completely will also be taken into account: one correct problem is better than two half-done problems.

\( \mathbb{Z}, \mathbb{Q}, \) and \( \mathbb{C} \) denote the integers, the rational numbers, and the complex numbers, respectively.

1. (a) Explain why there is a natural one-to-one correspondence between maximal ideals of \( \mathbb{C}[x] \) and the elements of \( \mathbb{C} \).

(b) Let \( R \) be a ring. Prove that if \( I \) is an ideal of \( R \) and \( I \neq R \), then \( I \) is contained in a maximal ideal of \( R \). If \( R \) is a commutative ring with 1. Let \( P \) be a prime ideal of \( R \). Prove that if there are ideals \( I_1, I_2, \ldots, I_n \), such that \( P = I_1 \cap I_2 \cap \ldots \cap I_n \), then \( P = I_j \) for some \( j \).

(c) Give an example of a commutative ring \( R \), an \( R \)-module \( M \), and an exact sequence of \( R \)-modules \( 0 \rightarrow A \rightarrow B \rightarrow M \) such that \( 0 \rightarrow A \otimes M \rightarrow B \otimes M \) is not exact.

(d) Suppose \( A \) is a hermitian (self-adjoint) matrix over the complex numbers. Prove that there is a matrix \( B \) such that \( A = B^2 \).

(e) Identify the \( \mathbb{Z} \)-module \( \mathbb{Z}[(1 + \sqrt{5})/2] / \mathbb{Z}[\sqrt{5}] \) as a standard finitely generated module over the PID \( \mathbb{Z} \).

2. Let \( R \) be a commutative ring with 1. Let \( P \) be a prime ideal of \( R \). Prove that if there are ideals \( I_1, I_2, \ldots, I_n \), such that \( P = I_1 \cap I_2 \cap \ldots \cap I_n \), then \( P = I_j \) for some \( j \).

3. Let \( p \) be a prime and let \( n \) be a natural number. Let \( \text{GF}(p^n) \) denote the field of order \( p^n \). Prove that the group of automorphisms of \( \text{GF}(p^n) \) is cyclic of order \( n \).

4. Suppose that \( G \) is a group of order 18. Prove that either \( G \) is abelian, or \( G \) is isomorphic to the dihedral group \( D_9 \), or \( G \) is generated by three elements \( a, b, c \), such that \( a^3 = b^3 = c^2 = 1 \), \( ab = ba \) and \( cac^{-1} = a^qb^r, bcb^{-1} = a^sb^t \), where \( \begin{bmatrix} q & r \\ s & t \end{bmatrix} \) is in \( \text{GL}_2(\mathbb{Z}/3\mathbb{Z}) \) and has order 2. (If you have time at the end of the test: how many non-isomorphic groups of the latter type are there?)

5. Find a set of matrices over the complex numbers such that any matrix (over \( \mathbb{C} \)) whose characteristic polynomial equals \( (x - 2)^3 \) is similar (conjugate) to one and only one matrix in your set. Prove your answer.
6. (a) Find the order of the group $\text{SL}_2(\mathbb{Z}/7\mathbb{Z})$ and prove your answer.
   (b) How many Sylow 7-subgroups does $\text{SL}_2(\mathbb{Z}/7\mathbb{Z})$ have? Find one explicitly.

7. Let $\alpha \in \mathbb{C}$ be a root of $x^3 + 2x + 2$.
   (a) Prove that $\mathbb{Q}[\alpha]$ is a field.
   (b) Find $(\alpha^2 + 1)^{-1}$ as a polynomial in $\alpha$.

8. Let $p$ be an odd prime. Let $F$ be splitting field of $x^p - 1$ over $\mathbb{Q}$. Prove that there is a unique field $K$ between $\mathbb{Q}$ and $F$ which is of degree 2 over $\mathbb{Q}$. Describe this field explicitly when $p = 5$. 