Ph.D. Preliminary Examination, March 1997
(Solve any 5 problems completely.)

1. Let \( \{X_n\} \) be a sequence of independent random variables.
   (a) If \( EX_n = 0 \) for \( n = 1, 2, \ldots \), and \( \sum_{n=1}^{\infty} \text{var}(X_n) < \infty \), show that \( \sum_{n=1}^{\infty} X_n \) converges a.s.
   (b) State (without proof) Levy's inequality and use it to prove that \( S_n = \sum_{k=1}^{n} X_k \) converges a.s. if and only if it converges in probability.

2. (a) Prove that for any r.v. \( X \)
   \[
   E|X| = \int_{0}^{\infty} P(|X| \geq t)dt.
   \]
   (b) Given a square integrable r.v. \( X \), show that for \( \lambda \geq 0 \),
   \[
   P(X - EX \geq \lambda) \leq \frac{\sigma^2(X)}{\sigma^2(X) + \lambda^2}.
   \]

3. (a) State (without proof) the Levy continuity theorem regarding a sequence of characteristic functions.
   (b) Let \( \{X_n\} \) be iid r.v.s with distribution \( F(x) \) having finite mean \( \mu \) and variance \( \sigma^2 \).
   Let \( S_n = X_1 + \cdots + X_n \). Show that
   \[
   \frac{S_n - n\mu}{\sigma \sqrt{n}} \rightarrow N(0, 1) \text{ in distribution as } n \rightarrow \infty.
   \]

4. (a) State (without proof) the Doob's maximum inequality and Kolmogorov's inequality.
   (b) Let \( F_n \) be a family of \( \sigma \)-algebras such that
   \[
   \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots
   \]
   and \( X \) be an integrable random variable. Show that
   \[
   E[X|F_n] \rightarrow E[X|\mathcal{F}_\infty] \text{ a.s. and in } L^1,
   \]
   where \( \mathcal{F}_\infty = \cap_{n=1}^{\infty} \mathcal{F}_n \).

5. If \( \{X_n\} \) are iid r.v.s, then \( E|X_1| < \infty \) if and only if \( \sum_{n=1}^{\infty} X_n \text{ sinn} \) converges a.s. for every \( t \in (-\infty, \infty) \).

6. Let \( \{X_n\} \) be iid r.v.s. Then,
   (a) \( n^{-1} \max_{1 \leq i \leq n} |X_i| \rightarrow 0 \) in probability if and only if \( n P(|X_1| > n) = o(1) \).
   (b) \( n^{-1} \max_{1 \leq i \leq n} |X_i| \rightarrow 0 \) a.s. if and only if \( E|X_1| < \infty \).

7. (a) Given a random variable \( X \) with finite mean square. Let \( \mathcal{D} \) be a \( \sigma \)-algebra. Show that \( E[X|\mathcal{D}] \) is the minimizer of \( E(X - \xi)^2 \) over all \( \mathcal{D} \)-measurable r.v.s \( \xi \), i.e.,
   \[
   E(X - E[X|\mathcal{D}])^2 \leq E(X - \xi)^2
   \]
   for all \( \mathcal{D} \)-measurable r.v.s \( \xi \).
   (b) Let \( (\Omega, \mathcal{F}, P) \) denote a probability space. Suppose \( f : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R} \) is a bounded \( \mathcal{B} (\mathbb{R}^n) \times \mathcal{C} \) measurable function and \( X \) be a \( n \)-dimensional \( \mathcal{D} \) measurable random variable. Assume \( \mathcal{C} \) and \( \mathcal{D} \) are independent. If \( g(x) := E f(x, \omega) \), then
   \[
   g(X) = E[f(X, \omega)|\mathcal{D}], \text{ a.s.}
   \]