PhD PRELIM in ALGEBRA

May 9, 1994

Put you name on all of your pages. Answer as many problems as you can, in as much detail as you can.

1. (a) State the Sylow Theorems (all "three parts").
   (b) Prove that no group of order 56 is simple.

2. Let G be the group of "rigid motions" of a regular tetrahedron $\triangle$; (you may assume $\triangle$ is centered at the origin of $\mathbb{R}^3$ and G is the group of rotations which carry $\triangle$ into itself). Prove that $G \cong A_4$ by considering the action of G on a suitable set.

3. Let R be a commutative Noetherian ring with identity, and prove that the polynomial ring $R[x]$ is Noetherian.

4. (a) Sketch carefully a proof of the fundamental theorem on the structure of finitely generated abelian groups.
   (b) Which pairs (if any) of the following additive abelian groups are isomorphic:
   $$\mathbb{Z}_{12} \times \mathbb{Z}_{90},\ \mathbb{Z}_{24} \times \mathbb{Z}_{45},\ \mathbb{Z}_{30} \times \mathbb{Z}_{36}?$$

5. (a) If g is an irreducible polynomial of degree $d > 1$ over a field F, prove there is a field extension E of degree d over F, in which g has at least one root.
   (b) If p is a prime number, and r a natural number, prove there exists a field F with precisely p^r elements. Is F unique?

6. (a) Give an example of a finite field extension of $\mathbb{Q}$ with Galois group S_3 and explain why in as much detail as you can.
   (b) Sketch how to construct a field extension of E/F with any finite group G as a Galois group. Quote any big theorems you need for the proof.

7. Let T be a Hermitian operator on a complex inner product space V.
   (a) If w is an eigenvector of T, prove $w^{\perp}$ is a T-invariant subspace of V.
   (b) If V has finite dimension, use a) to prove that V has an orthonormal basis consisting of eigenvectors of T.

8. Let the linear operator $L : V \to V$ define an action of $\mathbb{R}[x]$ on V as usual, and assume that V decomposes as the direct sum $\mathbb{R}[x]/p_1 \oplus \mathbb{R}[x]/p_2$, where $p_1 = x^2 + 3$, and $p_2 = (x^2 + 3)(x - 2)$. Find the a) minimal polynomial, b) characteristic polynomial, c) determinant, and d) rational canonical form, of L.