Algebra Preliminary Examination

May 10, 1993

Work all 8 problems.

1. Define a normal subgroup N of a group G and the quotient group G/N. State and prove the Fundamental Homomorphism Theorem (characterizing when a group homomorphism $G \to H$ factors through the quotient homomorphism $G \to G/N$). Also formulate the corresponding Fundamental Homomorphism Theorem for rings.

2. State the Sylow theorems (all parts) and describe all the Sylow subgroups of the alternating group A_5.

3. State the (finite dimensional) spectral theorem over the reals $\mathbb{R}$. Describe the procedure for diagonalizing the quadratic form $x^2 + y^2 + z^2 - xy - yz$ over $\mathbb{R}$. (You do not have to explicitly carry out each step.)

4. Define what it means for a square matrix over a field to be in Jordan form, and state the result on existence of a Jordan canonical form for square matrices. Find the Jordan canonical form of the following matrix. Find a basis with respect to which the matrix will be in Jordan canonical form. Also indicate the rational canonical form of the matrix.
$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 0 & -4 & 2 \end{pmatrix}$$

5. Define what it means for a finite field extension of fields, E over F, to be normal. Show that a tower of two finite normal extensions need not be normal. Prove that any finite extension of a finite field $\mathbb{F}_q$ is normal.

6. State the Fundamental Theorem of Galois Theory and illustrate it completely for the splitting field of the polynomial $x^8 - 1$ over the rationals $\mathbb{Q}$. Also indicate the structure of the Galois group of $x^8 - 2$ over $\mathbb{Q}$.

7. Let R be a ring with 1, let $A \to B \to C \to 0$ be an exact sequence of left R-modules, and let M be a right R-module. Prove that the induced sequence $M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is exact.

8. Let A be a commutative ring with 1 and let M be a finitely generated A-module. Prove that for each prime ideal p of A, the localization M_p (of M with respect to the multiplicative set $A - p$) is nonzero if and only if p contains the annihilator $ann(M) = \{a \in A | a \cdot m = 0 \text{ for all } m \in M\}$. Show that the "if and only if" need not hold if M is not assumed to be finitely generated.