Real Analysis Qualifying Exam

August, 2012

Give clear reasoning. State clearly which theorems you are using. You should not cite anything else: examples, exercises, or problems. Cross out material you do not want to be graded. Read through all the problems, do them in any order, the one you feel most confident about first. They are not in the order of difficulty.

Notation: $\mathbb{R}$ is the set of real numbers; $\mathbb{N}$ is the set of natural numbers; m denotes Lebesgue measure.

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x\to\pm\infty} f(x) = 0$. Prove that f is uniformly continuous.

2. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of Lebesgue measurable functions, and set

$$E := \{x \in \mathbb{R} : \text{ the numerical sequence } (f_n(x))_{n\in\mathbb{N}} \text{ converges } \}.$$

Prove that E is a Lebesgue measurable set.

3. Find the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx$$

4. Suppose f is a measurable finite-valued function on $[0, 1]$ and define
$g : [0, 1] \times [0, 1] \to [0, \infty)$ by $g(x, y) := |f(x) - f(y)|$.
Prove that if g is integrable on $[0, 1] \times [0, 1]$, then f is integrable on $[0, 1]$.

5. Let $f : [0, 1] \to \mathbb{R}$ by $f(x) := x^{-\frac{3}{5}}$.

(1) For which $p > 1$ does f belong to $L_p[0, 1]$?

(2) Which $q > 1$ have the property that fg is integrable for each function $g \in L_q[0, 1]$?

<u>Hint</u>: The function $h(x) = \frac{1}{x^{\frac{2}{5}}(1-\ln|x|)}$ will come in handy in the last part.

6. Write $C[0, 1]$ for the space of continuous real-valued functions on $[0, 1]$.

(1) Prove that $C[0, 1]$ is complete under the norm $||f||_\infty := \max_{x\in[0,1]} |f(x)|$.

(2) Prove that $C[0, 1]$ is not complete under the norm $||f||_1 := \int_0^1 |f(x)|\, dx$.