1. Let \( f \) be a differentiable function on \([a, b]\). We say that \( f \) is **uniformly differentiable** on \([a, b]\) if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that
\[
\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon
\]
whenever \( |x - y| < \delta \) with \( x, y \in [a, b] \).

(a) Prove that \( f \) is uniformly differentiable on \([a, b]\) if and only if \( f' \) is continuous on \([a, b]\).

(b) Give an example of a function that is differentiable on \([a, b]\) but fails to be uniformly differentiable on \([a, b]\) (no proofs required).

2. (a) Let \( f : [0, 1] \to \mathbb{R} \). Give a definition of what it means to say that \( f \) is a Lebesgue measurable function.

(b) Let \( \{f_k\} \) be a sequence of finite-valued measurable functions on \([0,1]\). Prove that
\[
\limsup_{k \to \infty} f_k
\]
is a measurable function.

3. Let \( \{f_k\} \) be a sequence of Lebesgue integrable functions on \( \mathbb{R} \). Recall that \( \{f_k\} \) is said to **converge in measure** to 0 if for every \( \varepsilon > 0 \),
\[
\lim_{k \to \infty} m(\{x \in \mathbb{R} : |f_k(x)| \geq \varepsilon\}) = 0,
\]
where \( m \) stands for Lebesgue measure on \( \mathbb{R} \).

(a) Give an example to illustrate the fact that \( f_k \to 0 \) in measure does not necessarily imply \( f_k \to 0 \) in \( L^1 \).

(b) Prove that if we make the additional assumption that there exists an integrable function \( g \) such that \( |f_k| \leq g \) for all \( k \), then \( f_k \to 0 \) in measure does imply that \( f_k \to 0 \) in \( L^1 \).

4. Let \( \varphi \in L^1(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 1 \) and \( \varphi_t(x) := t^{-n} \varphi(t^{-1}x) \). Prove that if \( f \in L^1(\mathbb{R}^n) \), then
\[
\lim_{t \to 0} \int_{\mathbb{R}^n} |f * \varphi_t(x) - f(x)| \, dx = 0,
\]
where
\[
f * \varphi_t(x) = \int_{\mathbb{R}^n} f(x - y) \varphi_t(y) \, dy
\]
denotes the convolution of \( f \) with \( \varphi_t \).

5. For each \( 1 \leq p \leq \infty \), define \( \Lambda_p : L^p([0,1]) \to \mathbb{R} \) by
\[
\Lambda_p(f) = \int_0^1 x^2 f(x) \, dx.
\]
Explain why \( \Lambda_p \) is a bounded linear functional and compute its norm (in terms of \( p \)).